SMITHSONIAN CONTRIBUTIONS TO KNOWLEDGE.

ON THE RELATIVE INTENSITY

OF THE

HEAT AND LIGHT OF THE SUN

UPON DIFFERENT

LATITUDES OF THE EARTH.

BY

L. W. MEECH, A. M.

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COMMISSION

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INTRODUCTION.

THE regular and almost uniform variations which meteorological tables exhibit, indicate a periodical cause of change, which evidently resides in the sun. The inquiry then arises, may not these variations be determined by theory from the apparent course of the sun? The first part of the present investigation thus suggested by inspection of monthly temperatures, was published in *Silliman's Journal of Science* for 1850. Since then, considerable extensions have been made, including expressions for annual values; a view of the whole of which is given in the following pages. At some future time, the researches may be resumed in another series.

The object of the investigation here presented, is to resolve the problem of solar heat and light, to the extent of the principle, that the intensity of the sun's rays, like gravitation, varies inversely as the square of the distance, without resorting to any other hypothesis. The principle is but a geometrical consequence of the divergence of the rays. This elementary view thus presents the sun shining upon a distant planet, and indicates the sum of the intensities received at the planet's surface in all its various phases of position and inclination.

In relation to the earth especially, the sum of the intensities must be referred to the exterior limit of the atmosphere which surrounds the globe. This condition, which is perhaps necessary in the present state of science, has the advantage of rendering the formulas as rigorously accurate as are the propositions of geometry and the conic sections.

Poisson, in 1835, observed that, "for the completion of the theory of heat, it is necessary that it should comprise the determination of the movements produced in aeriform fluids, in liquids and even in solid bodies; but geometers have not yet resolved this order of questions, of great difficulty, with which are connected the phenomena of the trade-winds, of certain currents observed in the sea, and the diurnal variations of the barometer." The subject is believed to be now included among the prize questions of the French Academy, and in the increasing number of researches, it is hoped that its difficulties may at length be effectively obviated.

The laws of Solar Intensity here derived *à priori*, have a general accordance with physical phenomena, and will furnish instructive comparisons with analogous values obtained by meteorological observations. The changes of the sun's intensity upon the inaccessible regions of the Pole will be included, to which the late

Arctic explorations have given unusual interest. And, among other advantages, light will be thrown upon geological researches relating to changes of the heat of the globe at very remote epochs.

It will be proper also to observe, that the method of summation, of which examples are given in the fifth and ninth sections, is more simple and direct than the process of discontinuous functions. The general reader, however, passing over the algebraic analysis, which is but a means for carrying out the leading conception already stated, will find the conclusions which flow from it plainly discussed in the remaining paragraphs of the several sections, and illustrated by tables and the accompanying curves.

At the close, the course of investigation has led to the development of a peculiar inequality in the annual duration of sunlight. The like series of values for the duration of twilight is also new, and will not be devoid of interest. But the main design has been—distinguishing between the sun's intensity and terrestrial temperatures—to carry out one comprehensive principle, by which the laws of the sun's intensity of heat and light are obtained to some degree of completeness, as a system, embracing the following topics in order:—

SECTION I. *Irradiated Surface upon the Planets.*—Zone of Differential Radiation; its Breadth and Area; its Extension by Refraction; its Changes of Position.

SECTION II. *The Sun's Intensity upon the Planets, in relation to their Orbits.*—Intensity proportional to the true longitude described. Table of Relative Intensity in equal times and in entire revolutions. Resemblance of the Earth to the planet Mars. Equality of Intensities during the four Seasons.

SECTION III. *Law of the Sun's Intensity at any Instant during the day.*—It is proportional to the length of a perpendicular line from the Sun's Centre to the Horizon. The Atmosphere. Causes of Climate.

SECTION IV. *The Sun's Diurnal Intensity.*—It depends on the Latitude, the Sun's Declination, Hour-angle, and Distance. Intensity upon the North Pole, during Summer, greater than upon the Equator. Graphical comparison of Intensities with Temperatures. Average Rate of Solar Intensity per hour. Retardation of the effects of the Sun's Intensity. Indication of Equatorial, Tropical, and Polar Calms.

SECTION V. *The Sun's Annual Intensity.*—Formula for the Summation of Series demonstrated. The Annual Intensity is measured by three Elliptic Functions. Tabular Values. Annual Intensity upon the Polar Circle equal to one-half of that upon the Equator. Analogy with the line of perpetual Snow. Graphical comparison of annual Intensities with annual Temperatures.

SECTION VI. *Average Annual Intensity upon a part or the whole of the Earth's Surface.*

SECTION VII. *Secular Changes of Intensity.*—Spots on the Sun's Disk. Leverrier's secular values of the Eccentricity connected with slight changes of Intensity. Tabular differences of annual Intensity 10,000 years ago from the present amount. Intensity during Summer and Winter influenced by the place of the Earth's Perihelion. Change of Intensity since the time of Hipparchus, 128 B. C. Conclusion that great Geological Changes must be referred to other causes than the Secular Inequalities of the Earth's Orbit. A probable result of the motion of the whole Solar System in space.

SECTION VIII. *Local and Climatic Changes.*—More equable Intensity in the Northern Hemisphere. A slight local inequality produced by daily change of the Sun's Declination. Of the Maximum and Minimum, or mid-summer and mid-winter Intensity. Climate of the Pole. The question of an open Arctic Sea.

SECTION IX. *Duration of Sunlight and Twilight.*—Perturbation of the annual Duration of Sunlight—its Epochs—its Analytic Expression. Of Civil and Astronomic Twilight. The Twilight Bow. Height of the Atmosphere calculated from Twilight. Limits of Twilight. Formulas for its Annual Duration. Tables of the Diurnal and Annual Values for the Northern Hemisphere. Delineations by Geometrical Curves.

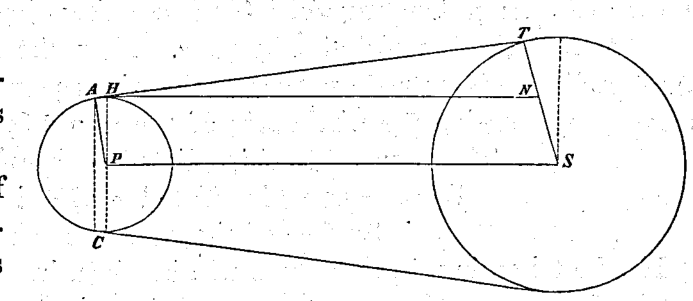
ON THE

RELATIVE INTENSITY OF THE HEAT AND LIGHT OF THE SUN.

SECTION I.

ON THE PROPORTION OF A PLANET'S SURFACE WHICH IS IRRADIATED BY THE SUN AT ANY GIVEN TIME.

It is evident that the extreme rays proceeding from the sun to the planet are *tangent* to the two spheres, as shown in the annexed diagram; where S denotes the centre of the sun, and P that of the planet.



Let $PS = \rho$, the radius-vector, or distance of the planet's centre from that of the sun.

Let $ST = R$, the radius of the sun, and $PA = r$, the radius of the planet, regarded as a sphere.

Through P, let a plane be drawn perpendicular to PS, and dividing the planet's surface into two equal hemispheres. The sun, being the greater body, illuminates not only the adjacent hemisphere of the planet, but also the zone or belt, AC, lying beyond; which may be called *the Zone of differential radiation.*

Let the angular breadth of this zone $APH = z$, and, drawing AN or ρ parallel to PS, the angle TAN is obviously equal to APH or z, since the including sides of the one angle are respectively perpendicular to those of the other, and, therefore, have the same relative inclination. Then, in the triangle ATN, which is right angled at T, by the condition of tangency,

$$sin\ TAN = \frac{NT}{AN}, \quad \text{or } sin\ z = \frac{R - r}{\rho}. \qquad (1.)$$

That is, *the sine of the angular breadth of the zone of differential radiation is equal to the difference of the radii of the sun and planet divided by the radius-vector of the planet's orbit.*

To express this value in another form, let A denote the semi-transverse axis of the planet's orbit, or its mean distance from the sun; let e denote the ratio of eccentricity, and θ the true anomaly estimated from the perihelion; then, by Analytical Geometry, $\rho = \frac{A\,(1 - e^2)}{1 + e \cos \theta}$; and hence,

$$\sin z = \frac{(R - r)\,(1 + e \cos \theta)}{A\,(1 - e^2)}. \qquad (2.)$$

Here for special values, making $\cos \theta$ successively equal to $-e$, $+1$, -1, and cancelling factors, we obtain for the values of $\sin z$ in order:—

$$\left.\begin{array}{ll} \text{Average,} & \sin z = \frac{R - r}{A}. \\ \text{Maximum,} & \sin z = \frac{R - r}{A\,(1 - e)}. \\ \text{Minimum,} & \sin z = \frac{R - r}{A\,(1 + e)}. \end{array}\right\} \qquad (3.)$$

Again, taking the length of arc z in the circle whose radius is 1, the breadth of the differential zone upon the planet will be rz; but since, for all the planets, z is less than 1°, its *sine* may be substituted from either of the former equations, and the same value essentially is represented by

$$\text{Linear breadth of zone} = r \sin z. \qquad (4.)$$

It is also proved in Geometry that the surface of a sphere whose radius is r, is equal to $4\,r^2 \pi$; π denoting 3.141592; and that the surface of a spherical zone is equal to its altitude multiplied by $2\,r\,\pi$. Now, the altitude of the zone of differential radiation is, in ratio to that of the whole planet, as $\sin z$ to 2, or $\frac{1}{2} \sin z$ to 1. Hence, representing the whole area of the planet by 1.

$$\text{The } \textit{proportion} \text{ of irradiated surface} = \tfrac{1}{2} + \tfrac{1}{2} \sin z. \qquad (5.)$$

$$\text{Whole surface irradiated} = (\tfrac{1}{2} + \tfrac{1}{2} \sin z)\,4\,r^2 \pi. \qquad (6.)$$

$$\text{Surface of the zone} = 2\,r^2 \pi \sin z. \qquad (7.)$$

If r be taken in miles, the area will be given in square miles.

The following table exhibits some of the primary phases of solar intensity upon the planets; and was obtained by substituting the proper astronomic elements in formulæ (3), (4), and (5).

PLANET.	Average breadth of zone.	Greatest breadth of zone.	Least breadth of zone.	Proportion of surface irradiated.
	Miles.	Miles.	Miles.	
Mercury	17.89	22.32	14.96	.505991
Venus	61.12	61.54	60.70	.503190
Earth	18.29	18.60	17.98	.500231
Mars	6.42	7.07	5.87	.500152
Vesta	.26	.28	.24	.500980
Jupiter	34.87	36.62	33.28	.500404
Saturn	18.17	19.25	17.21	.500222
Uranus	4.01	4.20	3.83	.500117
Neptune	6.14	6.19	6.08	.500087

In obtaining these tabular results, the earth's mean distance from the sun was taken at 95,273,870 miles, and its radius at 3,962 miles.

It will be perceived that the vast *magnitude* of the sun brings advantages of temperature and sunlight similar to those which the preponderance of its *mass* gives to the steadiness and uniformity of the planetary revolutions. Were the same amount of heat and light, radiated from a smaller body like the Moon, the effects would be restricted to a smaller portion of the Earth's surface; and the zone of differential radiation would be reversed to one of cold and darkness. But in the present beneficent arrangement, light and heat preponderate, counteracting extremes of heat and cold with a warmer temperature. And this effect is further prolonged by atmospheric refraction and reflection of the rays, which, rendering the transitions more mild and gradual, lessens the reign of night.

To estimate this effect of the *Refraction of Light*, we have only to find two points on the spherical surface of the earth, at such distance that the inclination of the two tangent rays from the Sun falling on them, shall be just equal to the horizontal refraction. The terrestrial radii drawn to these points will evidently be inclined at the same angle as their tangents, which is 34′ nearly, or 40 English miles. Thus it appears that the effect of refraction in widening the irradiated zone of the earth is more than twice as great as that arising from the apparent semi-diameter, or the mere size of the sun. Uniting the two effects, the sun is found to illuminate more than half the Earth's surface by a belt or zone that is 58 miles in width, encircling the seas and continents of the globe.

The advantage of the vast size of the sun is most conspicuous upon the planet Venus, our evening and morning star, where the belt of illumination is sixty-one miles in width, as shown in the preceding table. The next in rank is Jupiter, whose belt of greater illumination is thirty-five miles wide; while those of Mercury, the Earth, and Saturn, are nearly eighteen miles in breadth. In the last column of the table, it will be observed that the asteroid Vesta, though situated beyond Mars, yet has, in consequence of its smaller size, a greater proportion of illuminated surface than the Earth.

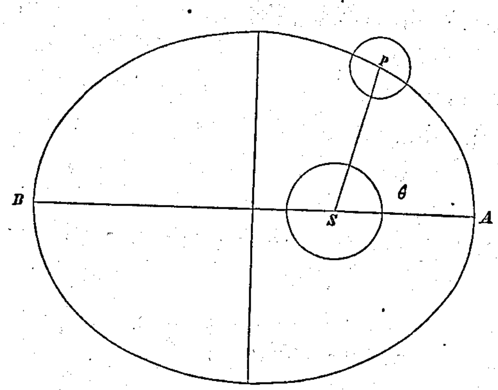
From formula (7), it is found that the zone of differential illumination upon the Earth extends over 455,400 square miles; or, including the additional area due to 34′ horizontal refraction, it comprehends an aggregate of 1,430,800 square miles of surface. The position of this great zone is continually changing, and in turn it overspreads every island, sea, and continent. At the vernal equinox, when the Sun is vertical to the Equator, it will readily be perceived that the larger base of this zone is a great circle passing through the Poles and having the Earth's axis for its diameter. From this position it gradually diverges, till at the summer solstice, one extremity of its diameter will be in the Arctic, and the other in the Antarctic circle. Thence it gradually returns to its former position at the Poles at the autumnal equinox, all the while revolving like a fringed circle around the globe, and accompanied with the lustrous tints and shadows which variegate the dawn and close of day.

SECTION II.

LAW OF THE SUN'S INTENSITY UPON THE PLANETS IN RELATION TO THEIR ORBITS.

THE preceding Section represents the Sun's action upon a distant planet at a given distance, or at rest. It is here proposed to examine the effect when the distance is variable; that is, supposing the planet to commence its motion from a state of rest, in an elliptical orbit, to determine the intensity received during its passage through any part, or the whole of its orbit.

In the annexed figure, let S denote the Sun situated in one focus; P the Planet's position at a given time; A, the perihelion or point in the orbit nearest the sun, and B, the aphelion or point farthest from the sun.



Let SP or ρ denote the radius-vector; ASP or θ, the true anomaly; e, the ratio of eccentricity; and $a+nt$, the mean anomaly; n, being the mean motion in the unit of time.

If A denote the semi-transverse axis, it is well known that $A^2\pi\sqrt{1-e^2}$ will express the whole area of the ellipse, and $\int \frac{1}{2}\rho^2 d\theta$, the area of the elliptic sector corresponding to θ, where π denotes 3.14 1592, or a semi-circumference. Hence by Kepler's law, that equal areas are described by the radius-vector in equal times,

$$A^2\pi\sqrt{1-e^2} : \int \tfrac{1}{2}\rho^2 d\theta :: 2\pi : a+nt.$$

Reducing to an equation and differentiating,

$$\frac{1}{\rho^2} = \frac{d\theta}{A^2 n dt\sqrt{1-e^2}}. \qquad (8.)$$

Since heat and light vary inversely as the square of the distance ρ, the second member evidently measures their intensity at any instant. Then, as pointed out in the Calculus, we may regard the second member as the ordinate, and the time t as the abscissa of a curve. Multiplying the equation by dt, therefore, and integrating between the limits of any two anomalies, θ and θ', we obtain for the sum of the intensities,

$$\int \frac{1}{\rho^2} dt = \frac{\theta - \theta'}{A^2 n\sqrt{1-e^2}}. \qquad (9.)$$

In interpreting this result, we know that the orbitual motion of a planet is not uniform, being accelerated in perihelion and retarded in aphelion. Hence, in the annual variations of radius-vector, the Earth does not receive equal increments of heat and light in equal times; but *the amount received in any given interval, is exactly proportional to the true anomaly or true Longitude described in that interval.*

This important law appears to have been first published in the Pyrometry of Lambert.

This point being established, let us, in the next place, compare the intensities received by the Planets during entire revolutions in their orbits. In the preceding formula, making $\theta - \theta'$ equal to an entire circumference, the sum of the intensities during a complete revolution, is found to be $u = \frac{2\pi}{A^2 n \sqrt{1-e^2}}$. Let this refer to the earth, and accenting the values for any other planet, $u' = \frac{2\pi}{A'^2 n' \sqrt{1-e'^2}}$. Now n, n', being inversely proportional to the planets' periodic times, we have by the third law of Kepler, $n^2 : n'^2 :: A'^3 : A^3$, or $n A^{\frac{3}{2}} = n' A'^{\frac{3}{2}}$. Whence by substitution and division, we obtain for the *relative intensity* upon any planet *in an entire revolution*,

$$\frac{u'}{u} = \frac{\sqrt{A(1-e^2)}}{\sqrt{A'(1-e'^2)}}. \qquad (10).$$

In like manner, the ratio of intensity *for equal times*, depending simply on the inverse square of the distances, will be represented by

$$\frac{u'}{u} = \frac{A^2}{A'^2}. \qquad (11).$$

With these last two formulas, the following table has been prepared from the usual astronomic elements:—

The Sun's Relative Intensity upon the Principal Planets.

Planet.	In a whole revolution.	In equal times.		
		Mean Distance.	Perihelion.	Aphelion.
Mercury	1.643	6.677	10.573	4.592
Venus	1.176	1.911	1.937	1.885
Earth	1.000	1.000	1.034	0.967
Mars	.813	.431	0.524	0.360
Jupiter	.439	.037	.041	.034
Saturn	.324	.011	.012	.010
Uranus	.228	.003	.003	.003
Neptune	.182	.001	.001	.001

It should be observed that the foregoing table does not take account of the different dimensions of the planets, but refers to a unit of plane surface upon their disks, which is exposed perpendicularly to the rays of the perpetual sun. Upon the disk of Mercury, the solar radiation appears to be nearly seven times greater than on the Earth; while upon Neptune, it is only as the one-thousandth part, in equal times. In entire revolutions, however, the intensities received will be seen to approach more nearly to equality.

The intensities are thus unequal; and, by a calculation founded on the apparent brightness of the planets as estimated by the eye, Prof. Gibbes has shown, in the Proceedings of the American Association for the Advancement of Science for 1850, that the reflective powers are also greater, according as the several planets are more distant from the Sun.

Another feature worthy of mention, is the resemblance of the earth to the planet Mars; upon which Sir W. Herschel has remarked: "The analogy between Mars and the Earth is, perhaps, by far the greatest in the whole solar system. The diurnal motion is nearly the same, the obliquity of their respective ecliptics not very different; of all the superior planets, the distance of Mars from the Sun is by far the nearest alike to that of the Earth; nor will the length of the Martial year appear very different from what we enjoy, when compared to the surprising duration of the years of Jupiter, Saturn, and Uranus. If we then find that the globe we inhabit has its polar region frozen and covered with mountains of ice and snow, that only partly melt when alternately exposed to the sun, I may well be permitted to surmise that the same causes may have the same effect on the globe of Mars; that the bright polar spots are owing to the vivid reflection of light from frozen regions; and that the reduction of those spots is to be ascribed to their being exposed to the sun."

Recurring now to equation (9) and the proposition following, it will readily be inferred that during each of the four astronomic seasons of Spring, Summer, Autumn, and Winter, the intensities received from the sun are precisely equal. For in each season, the earth passes over three signs of the zodiac, or a quadrant of longitude. The equality of intensities, however, applies to the entire globe regarded as one aggregate, and is consistent with local alternations, by which it is summer in the northern hemisphere when it is winter in the southern. Deferring the consideration of these local inequalities, however, we may here illustrate the connection of the seasons with the elliptic motion from an ephemeris. In the year 1855, for example, spring in the northern hemisphere, commencing at the vernal equinox March 20th, lasts eighty-nine days; summer, beginning at the summer solstice June 21, continues ninety-three days; autumn, commencing at the equinox, September 23, continues ninety-three days; and winter, beginning at the winter solstice, December 22, lasts ninety days; yet, notwithstanding their unequal lengths, the amounts of heat and light which the whole earth receives are equal in the several periods.[1]

At the present time the earth is in perihelion, or nearest the sun about the 1st of January, and farthest from the sun on the 4th day of July. A special cause must, therefore, be assigned for the striking fact which Professor Dove has shown by comparison of temperatures observed in opposite regions of the globe, namely: that the mean temperature of the habitable earth's surface in June considerably exceeds the temperature in December, although the earth in the latter month is nearer to the sun. This result is attributed by that meteorologist to the greater quantity of land in the northern hemisphere exposed to the rays of the sun at the summer solstice in June; while the ocean area has less power for this object, as it absorbs a large portion of the heat into its depths. Had land and water been equally distributed; in other words, were the earth a homogeneous sphere, the alleged inequality of temperature, it is obvious, would never have existed.

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[1] Since the earth is not strictly a sphere, but an oblate spheroid, it evidently presents its least section perpendicular to the rays of the sun at the equinoxes. As the sun's declination increases, the section also increases and attains its limit at the solstice. The variation, however, appears to be not material, and compensates itself in each season.

SECTION III.

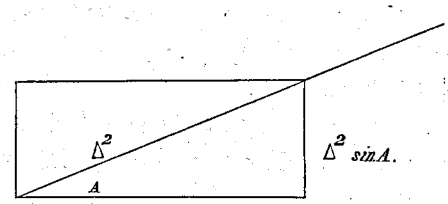
LAW OF THE SUN'S INTENSITY AT ANY INSTANT DURING THE DAY.

THE rays which emanate from the Sun's disk into space proceed in diverging lines in the same manner as if they issued directly from the centre. And, on arriving at the Earth, their intensity as before stated will be inversely proportional to the square of the distance.

But the more obvious phenomena of solar heat and light are manifested to us under a secondary law. The Sun's intensity first becomes sensible in the eastern rays of morning; it gradually increases to a maximum during the day; it declines on the approach of the shades of evening, and becomes discontinuous during the night. On the morning following the same course is renewed, and continued successively through the year. Ordinary sensation and experience lead us to associate the degree of solar heat at any part of the day, with the apparent height which the sun has then attained above the horizon. Indeed, theory determines that at four in the afternoon, or any other instant during the day, *the Sun's intensity is proportional to the length of a perpendicular line dropped from the Sun to the plane of the apparent horizon, or varies as the size of the sun's altitude.*

The reason of this secondary law will be understood by regarding the beam of solar rays which traverses in a line from the sun to the observer, to be resolved, according to the parallelogram of forces, into a horizontal and a vertical component. The horizontal component running parallel to the earth's surface is regarded as inoperative, while the vertical component measures the direct heating effect.

This relation is more fully shown in the annexed figure, where A denotes the sun's apparent altitude above the horizon. The sun's intensity or impulse in an oblique direction will be measured by the inverse square of the distance, or the direct square of the sun's apparent semi-diameter Δ. If, therefore, Δ^2 denotes the intensity of the rays in a straight line from the sun, $\Delta^2 \sin A$, *will be the vertical component or heating force of the rays.* And these terms being in ratio as 1 to $\sin A$, the latter component will be represented by a perpendicular line from the sun's centre to the horizon.



Instead of thus decomposing the intensity after the manner of a force in Mechanics, as first proposed by Halley, in 1693, the same law may be obtained in an entirely different way from the principle of the inverse square of the distance. The latter mode appears to present it in a more evident light, and was suggested in the original beginnings of the present investigation, which were published in Silliman's *Journal of Science* for the year 1850.

It proceeds as follows:—

Let L = the 'apparent' Latitude of the place,
D = the sun's meridian Declination,
Δ = the sun's apparent semi-diameter,
A = the sun's Altitude, and
H = the Hour-angle from noon.

Also in reference to future applications, let
T = the sun's true Longitude, and
ω = the obliquity of the Ecliptic.

The horizontal section of a cylindrical beam of rays from the Sun's disk upon a plain on the Earth's surface, is well known to be an ellipse; and if 1 denote the sun's radius, 1 will likewise denote the semi-conjugate axis of this projected ellipse; while the horizontal projection, $\frac{1}{\sin A}$, will be the semi-transverse axis. The area of the elliptic projection is, therefore, $1 \times \frac{1}{\sin A} \times \pi$. But the intensity of the same quantity of heat being inversely as the space over which it is diffused, the reciprocal of this area, or $\sin A$, on rejecting the constant π, will express the sun's heating effect, supposing the distance to be constant for the same day. But, on comparing one day with another, the intensity further varies inversely as the square of the distance, that is, directly as the square of the apparent diameter or semi-diameter of the disk. Hence, generally, $\Delta^2 \sin A$, expresses the sun's intensity at any given instant during the day.

To determine the value of $\sin A$, by spherical trigonometry, the sun's angular distance from the pole, or co-declination, the arc from the pole to the zenith, or co-latitude, and the included hour-angle from noon are given to find the third side or co-altitude. Writing, therefore, sines instead of the cosines of their complements,

$$\sin A = \sin L \sin D + \cos L \cos D \cos H.$$
$$\Delta^2 \sin A = \Delta^2 \sin L \sin D + \Delta^2 \cos L \cos D \cos H. \qquad (12.)$$

At the time of the equinoxes, D becomes 0, and the expression of the sun's intensity reduces to $\Delta^2 \cos L \cos H$. That is, the degree of intensity then decreases from the equator to each pole, and is *proportional to the cosine of the latitude.* At other times of the year, however, a different law of distribution prevails, as indicated by the formula.

The intensity at a fixed distance being as the sine of the altitude, it follows that the sun shining for sixteen hours from an altitude of 30°, would exert the same heating effect upon a plain, as when it shines during eight hours from the zenith; since *sin* 30° is 0.5, and *sin* 90° is 1. At least, such were the result independently of radiation.

By some writers, the measure of vertical intensity, as the sine of the sun's altitude, has been stated without limitation. Approximately it may apply at the habitable surface of the earth, when the influence of the atmosphere is neglected; yet it is strictly true only at the exterior of the atmospheric envelope which encompasses the globe, or at the outer limit where matter exerts its initial change upon the incident rays.

The distinction here explained has not only engaged the attention of the most eminent meteorologists of modern times, but was equally adopted in ancient philosophy, as appears in the following passage from Plato's Phædon, LVIII: "For around the earth are low shores, and diversified landscapes and mountains, to which are attracted water, the cloud, and air. But the earth, outwardly pure, floats in the pure heaven like the stars, in the medium which those who are accustomed to discourse on such things call ether. Of this ether, the things around are the sediment which always settles and collects upon the low places of the earth. We, therefore, who live in these terraqueous abodes, are concealed, as it were, and yet think we dwell above upon the earth. As one residing at the bottom of the sea might think he lived upon the surface, and, beholding the sun and stars through the water, might suppose the sea to be heaven. The case is similar, that through imperfection we cannot ascend to the highest part of the atmosphere, since, if one were to arrive upon its upper surface, or becoming winged, could reach there, he would on emerging look abroad, and, if nature enabled him to endure the sight, he would then perceive the true heaven and the true light."

In modern times, the researches of Poisson led him to the philosophic conclusion now generally received, that the highest strata of the air are deprived of elasticity by the intense cold; the density of the frozen air being extremely small, *Théorie de la Chaleur*, p. 460. An atmospheric column resting upon the sea may thus be regarded as an elastic fluid terminated by two liquids, one having an ordinary density and temperature, and the other a temperature and density excessively diminished.

Although the sun's intensity, which is here the subject of investigation, is the principal source of heat, yet its effects are modified by proximate causes of climate; of which, the following nine are enumerated by Malte Brun:—

1st.—Action of the sun upon the atmosphere.
2d.—The interior temperature of the globe.
3d.—The elevation above the level of the ocean.
4th.—The general inclination of the surface and its local exposure.
5th.—The position of mountains relative to the cardinal points of the compass.
6th.—The neighborhood of great seas and their relative situation.
7th.—The geological nature of the soil.
8th.—The degree of cultivation and of population to which a country has arrived.
9th.—The prevalent winds.

The same author observes, in relation to the fourth enumerated cause, that northeast situations are coldest; and southwest, warmest. For the rays of the morning which directly strike the hills exposed to the east, have to counteract the cold accumulated there during the night. The heat augments till three in the afternoon, when the rays fall direct upon southwest exposures, and no obstacle now prevents their utmost action.

SECTION IV.

DETERMINATION OF THE SUN'S HOURLY AND DIURNAL INTENSITY.

IN the last Section, the sun's vertical intensity upon a given point of the earth's surface at any instant during the day, was proved to be measured by a perpendicular drawn from the centre of the Sun to the plane of the horizon. If perpendiculars be thus let fall at every instant during an hour, the sum of the perpendiculars will evidently represent the sum of the vertical intensities received during the hour, which sum may be termed the Hourly Intensity.

The Integral Calculus furnishes a ready means of obtaining this sum. For during any one day, the sun's distance or apparent semi-diameter, and the meridian Declination, may be regarded as constant, while H alone varies, and the deviations from the implied time of the sun's rising and setting will compensate each other. Therefore, multiplying the equation of instantaneous intensity (12) by $d\,H$, since astronomy shows that H varies uniformly with the time, and integrating between the limits of any two hour angles, H', H'', we obtain an expression for the hourly intensity.

In like manner let H denote the semi-diurnal arc, and integrating between the limits 0 and H, we obtain the intensity for a half day, which, on cancelling the constant multiplier 2, may be taken for the whole day, or Diurnal Intensity, as follows:—

$$\int \Delta^2 \sin A\ d\,H = \Delta^2 H \sin L \sin D + \Delta^2 \cos L \cos D \sin H. \qquad (13.)$$

The diurnal intensity is, therefore, proportional to the product of the square of the sun's semi-diameter into the semi-diurnal arc, multiplied by the sine of the latitude into the sine of the sun's declination, *plus* the like product of the square of the sun's semi-diameter into the sine of the semi-diurnal arc multiplied by the cosine of the latitude into the cosine of the declination. This aggregate obviously changes from day to day, according to the sun's distance and declination.

Introducing the astronomic equation, $\cos H = -\tan L \tan D$, or in another form,

$$\cos L \cos D = -\frac{\sin L \sin D}{\cos H};$$ the expression reduces to the following:

$$\int \Delta^2 \sin A\ d\,H = \Delta^2 \sin L \sin D\,(H - \tan H).$$

It only remains to adopt a unit of intensity, the choice of which is entirely arbitrary. For the present, and in reference to Brewster's formula hereafter noticed, we will assume the intensity of a day on the equator at the time of the vernal equinox to be 81.5 units. For this case, where D and L are each 0, formula (13) reduces to Δ^2, which is $(965'')^2$; hence $81.5 \div (965'')^2$, or k, will be the multiplier for reducing all other values to the same scale; where the common logarithm of k is $\bar{5}.94210$. Denoting the annual intensity by u, and taking Δ in seconds of arc, we have in units of intensity,

$$u = k\,\Delta^2 \sin L \sin D\,(H - \tan H). \qquad (14.)$$

The following cases under the general formula may here be specified:—

First, at the time of the *Equinoxes*, D is 0, and consequently H is 6^h; substituting these values in (13) and converting into units,

$$u = k \Delta^2 \cos L. \qquad (15.)$$

Hence the sun's daily intensity for all places on the earth is then *proportional to the cosine of the latitude.* As the equinoxes in March and September lie intermediate between the extremes or maxima of heat and light in summer, and their minima in winter, the presumption naturally arises that the same expression will approximate to the mean annual intensity. The coincidence is accordingly worthy of note, that the best empirical expression now known for the annual temperature in degrees Fahrenheit, given by Sir David Brewster, in the *Edinburgh Philosophical Transactions*, Vol. IX, is $81.°5 \cos L$, being also proportional to the cosine of the latitude. It is remarkable that Fahrenheit, in 1720, should have adjusted his scale of temperature to such value, that this formula applies, without the addition of a constant term.

Secondly, for all places on the *Equator*, the latitude L is 0; and H is 6^h, or the sun rises and sets at six, the year round, exclusive of refraction. Consequently the Sun's diurnal intensity varies slowly from one day to another, being *proportional to the cosine of the meridian Declination*, or,

$$u' = k \Delta^2 \cos D. \qquad (16.)$$

Thirdly, at the *South or the North Pole*, the latitude L is 90°; and since $\tan 90°$ is infinite, the astronomic relation $\cos H = - \tan L \tan D$ is illusory, except when D is 0. The physical interpretation of this feature is, that at the North Pole, the sun rises only at the vernal equinox in March, and continues wholly above the horizon, till it sets at the autumnal equinox. Thus to either Pole, the sun rises but once, and sets but once in the whole year, giving nearly six months day, and six months night. Now suppose the six months day to be divided into equal portions of twenty-four hours each; then, in reference to formula (13), H is 12^h, and *the intensity during twenty-four hours of polar day is proportional to the sine of the Declination at the middle of the day;* or,

$$u'' = k \Delta^2 \pi \sin D.$$

This term varies much faster than the cotemporary value on the equator. And comparing the two expressions, it appears that during the summer season, in each twenty-four hours, the Sun's intensity upon the Equator is to that upon the Pole, in the following proportion:—

$$u' : u'' :: 1 : \pi \tan D. \qquad (17.)$$

Fourthly, at the summer solstice, when the intensity on the Pole is a maximum, D is 23° 28′, and the preceding ratio becomes as 1 to 1.25; or the Polar intensity is one-fourth part greater than on the Equator (Plate IV). The difference evidently arises from the fact that daylight in the one place lasts but twelve hours out of twenty-four, while at the Pole the sun shines on through the whole twenty-four hours.

It were interesting to find when this Polar excess begins and ends, which may be ascertained by equating the last two terms of (17). The condition $\pi \tan D = 1$, thus gives D equal to 17° 40′, which is the sun's Declination on May 10th, and again on August 3d. Therefore, *during this long interval of eighty-five days, comprehending nearly the whole season of summer, the Sun's vertical intensity over the North Pole is greater than upon the Equator.* To this subject we shall again recur in a subsequent Section.

Fifthly, having glanced at these particular cases of the formula, let a more complete survey be made for the northern hemisphere. And the same will equally apply to the southern hemisphere, allowing for the reversal of the seasons and change of the Sun's distance. In equation (14), when H exceeds 6^h, and when the declination D is south, a change of sign would be introduced; but the proper trigonometric signs will be observed simply by using the upper sign in summer, or when the declination is north, and the lower sign during the rest of the year, in the annexed formula of daily Intensity:—

$$u = [\bar{5}.94210]\, \Delta^2 \sin L \sin D\, (\tan H \pm H). \qquad (18.)$$

Here brackets include the logarithm of the co-efficient k; is to be taken in seconds of arc; H is the actual length of the semi-diurnal arc to radius 1, and *tan H* is the natural tangent. The subjoined table has been computed in this manner, for intervals of fifteen days, and expresses the results in *units of intensity*. In the last three columns for the Frigid Zone the braces include values for the days when the sun shines through the whole twenty-four hours; the blank spaces indicate periods of constant night.

The Sun's Diurnal Intensity at every Ten Degrees of Latitude in the Northern Hemisphere. (Plate I.)

A. D. 1853.	Lat. 0°.	Lat. 10°.	Lat. 20°.	Lat. 30°.	Lat. 40°.	Lat. 50°.	Lat. 60°.	Lat. 70°.	Lat. 80°.	Lat. 90°.
Jan. 1	77.1	67.2	55.8	42.8	30.1	16.5	5.1	...	...	...
" 16	78.1	68.9	58.2	45.8	32.7	19.3	7.2	...	...	...
" 31	79.6	71.7	61.9	49.7	38.6	25.0	11.9	1.4	...	...
Feb. 15	81.0	74.7	66.6	55.6	45.1	31.9	19.0	6.4	...	...
Mar. 2	81.6	78.0	71.3	62.9	52.7	41.1	27.9	14.5	2.1	...
" 17	82.0	80.2	76.0	69.6	61.1	50.2	37.1	25.5	11.6	...
April 1	80.8	81.4	79.5	75.3	68.9	60.2	49.9	38.0	25.6	20.5
" 16	79.0	81.7	82.0	79.5	75.1	68.6	61.1	51.4	44.0	44.6
May 1	76.9	81.5	83.7	83.6	80.8	77.1	70.9	64.6	64.3	65.3
" 16	74.7	80.8	84.7	86.7	85.7	83.3	79.7	76.8	80.3	81.5
" 31	73.0	80.1	85.1	87.8	88.9	87.8	85.7	86.8	91.0	92.4
June 15	72.0	79.6	85.2	88.4	90.1	89.9	88.8	91.7	96.1	97.6
July 1	72.0	79.5	85.0	88.5	90.4	89.5	88.4	90.8	95.1	96.6
" 16	73.0	79.8	84.7	87.5	87.6	86.5	84.1	84.3	88.3	89.7
" 31	74.7	80.4	83.9	85.1	84.5	81.6	77.3	73.4	76.2	77.4
Aug. 15	76.7	80.8	82.7	82.4	79.8	74.7	68.2	60.9	59.2	60.1
" 30	78.5	80.7	80.6	77.7	72.1	65.5	57.3	47.7	38.8	38.9
Sept. 14	79.8	79.8	77.5	72.6	65.6	58.8	46.9	34.5	21.9	14.7
" 29	80.5	78.4	73.8	67.0	57.8	47.0	36.2	22.5	9.0	...
Oct. 14	80.7	76.4	69.7	61.0	50.2	38.2	25.7	12.6	1.0	...
" 29	79.9	73.5	65.0	54.6	42.5	30.1	17.5	5.2	...	...
Nov. 13	78.8	70.7	60.8	49.8	37.1	23.8	11.0	0.9	...	...
" 28	77.5	68.3	57.3	45.3	31.8	18.9	6.8	...	...	...
Dec. 13	76.9	66.9	55.4	43.0	30.3	16.3	4.9	...	...	...

To indicate the law of the Sun's Diurnal Intensity to the eye also, I have taken the relative units in the table as ordinates, and their times for abscissas, and traced curves through the series of points thus determined, as shown in the accompanying diagram (Plate I).

The Equatorial curve will be observed to have two maxima at the Equinoxes in March and September, and two minima at the Solstices in June and December. Since the earth is nearer the sun in March than in September, the curve shows a greater intensity in the former month, other things being equal.

In the latitude of 10°, the Sun will not be vertical at the summer solstice, but only when the Declination is 10° N., which happens twice in the year. The curve corresponds in every particular with the known course of the sun. Above the latitude of 23° 28′, the tropical flexure entirely disappears; and there is only a single maximum at midsummer.

For comparison with the curves of *Intensity*, I have also traced curves of *Temperature* observed at Calcutta, in lat. 22° 33′ N.; at New Orleans, in lat. 29° 57′; at Philadelphia, in lat. 39° 57′; at London, in lat. 51° 31′; and at Stockholm, in lat. 59° 20′. The values for Stockholm represent the averages for every five days during fifty years, as given in the *Encyclopædia Metropolitana*, article Meteorology. The curve for Philadelphia is adjusted from the daily observations made at the Girard College Observatory from 1840 to 1845, under the direction of Prof. Bache. The rest are interpolated graphically from the mean monthly temperatures.

Retardation of the Effect.—In the Temperate Zone the Temperatures will be seen to attain their maximum about one month later than the sun's intensity would indicate. At Stockholm it is somewhat more than a month; and, during this interval the earth must receive, during the day, more heat than it loses at night; and, conversely, after the winter solstice, it loses more heat during the night than it receives by day. In illustration of this point, and to approximately verify the formula, I here insert a former computation of the sun's Intensity for the 15th day of each month, on the latitude of Mendon, Mass., and the results are found to agree very nearly with those observed at that place about *one month later*, as follows: (The observed values are taken from the *American Almanac* for 1849, and are derived from fifteen years' observations.)

Computed values.			Observed values.		Difference.
Jan. 15	5040	23°.3	24°.3	Feb. 15	+1°.0
Feb. "	7142	33°.1	33°.5	Mar. "	+ .4
Mar. "	9764	45°.2	45°.8	April "	+ .6
April "	12574	58°.3	55°.0	May "	—3°.3
May "	14482	67°.1	64°.5	June "	—2°.6
June "	15346	71°.1	71°.8	July "	+ .7
July "	15085	69°.9	68°.9	Aug. "	—1°.0
Aug. "	13437	62°.3	61°.0	Sept. "	—1°.3
Sept. "	10860	50°.3	48°.5	Oct. "	—1°.8
Oct. "	8080	37°.5	38°.9	Nov. "	+1°.4
Nov. "	5638	26°.1	27°.7	Dec. "	+1°.6
Dec. "	4510	20°.9	26°.0	Jan. "	+5°.1

It may be proper to observe that the formula was divided by *sin L*, a constant factor; and the numbers in the second column were then successively computed: their sum, divided by twelve, gave 10163 as the mean, to be compared with 47°.1, the observed mean at Mendon. Then as 10163 : 47°.1 :: 5040 : 20°.3, Jan. 15, etc.

Let it also be observed, that the Mendon values are the monthly means, which do not always fall on the 15th day, but nearly at that time.

Rate per Hour of the Sun's Intensity.—To glance at the subject from another point of view, let us consider the *Rate*, or the relative number of heating rays per hour. For any day, if we divide the computed Intensity by the length of the day, the quotient will express the average Hourly Intensity, denoted by R; thus,

$$R = \frac{u}{2H} = \tfrac{1}{2}\,[\bar{5}.94210]\,\Delta^2 \sin L \sin D \left(\frac{\tan H}{H} \pm 1\right). \qquad (19.)$$

In the accompanying table, the values of the rate R are exhibited at intervals of fifteen days, and for every ten degrees of latitude. From this, Plate II is constructed; and for comparison with the Daily Rate of Intensity, the Daily Range of the Thermometer is also delineated for Trinconomalee, on the coast of Ceylon (lat. 9° N.)—taking 5°.72 Fahr. plus $\frac{1}{5}$th of the mean daily ranges, as ordinates; also for Philadelphia (lat. 39° 57′), taking here 7° plus $\frac{1}{3}$rd of the daily ranges; for Göttingen (lat. 51° 32′), taking $\frac{1}{3}$rd of the daily ranges; and for Boothia Felix (lat. 70° N.), taking $\frac{5}{18}$ths of the daily ranges in degrees Fahrenheit as ordinates. These changes are arbitrary, but are analogous to the conversion of thermometric scales, and still preserve the original law of the curves. The peculiar inflexion at the vertex of the curve of Hourly Intensity for latitude 70°, evidently arises from the change to constant day. And apparently the hourly rates of Plate II, coincide more nearly with the temperatures of Plate I, than do the Diurnal Intensities, or absolute amounts.

Average Rate of the Sun's Hourly Intensity, or Relative Number of Vertical Rays per Hour.
(Plate II.)

A. D. 1853.	Lat. 0°.	Lat. 10°.	Lat. 20°.	Lat. 30°.	Lat. 40°.	Lat. 50°.	Lat. 60°.	Lat. 70°.	Lat. 80°.	Lat. 90°.
Jan. 1	6.43	5.89	5.16	4.24	3.26	2.08	0.88	...	...	...
" 16	6.51	5.99	5.32	4.44	3.44	2.32	1.12	...	...	...
" 31	6.63	6.20	5.56	4.66	3.86	2.75	1.56	0.34	...	...
Feb. 15	6.75	6.38	5.85	5.05	4.27	3.22	2.11	0.92	...	...
Mar. 2	6.80	6.59	6.11	5.50	4.71	3.78	2.70	1.56	0.35	...
" 17	6.83	6.70	6.38	5.85	5.15	4.25	3.17	2.21	1.03	...
April 1	6.73	6.71	6.50	6.09	5.51	4.73	3.82	2.76	1.64	0.86
" 16	6.58	6.67	6.56	6.21	5.70	5.02	4.24	3.22	1.83	1.86
May 1	6.40	6.59	6.57	6.33	5.86	5.32	4.50	3.52	2.68	2.72
" 16	6.23	6.48	6.53	6.40	6.01	5.46	4.71	3.55	3.35	3.40
" 31	6.08	6.39	6.49	6.36	6.07	5.54	4.78	3.62	3.79	3.85
June 15	6.00	6.33	6.45	6.34	6.07	5.57	4.81	3.82	4.00	4.07
July 1	6.01	6.32	6.44	6.36	6.12	5.58	4.83	3.78	3.96	4.03
" 16	6.08	6.38	6.46	6.37	6.01	5.51	4.75	3.51	3.68	3.74
" 31	6.22	6.46	6.50	6.32	5.98	5.42	4.65	3.55	3.18	3.22
Aug. 15	6.39	6.56	6.50	6.30	5.87	5.23	4.38	3.43	2.47	2.50
" 30	6.54	6.60	6.48	6.12	5.53	4.87	4.05	3.10	1.90	1.62
Sept. 14	6.64	6.60	6.37	5.92	5.45	4.70	3.68	2.61	1.50	0.61
" 29	6.70	6.57	6.21	5.68	4.93	4.05	3.16	2.04	0.89	...
Oct. 14	6.73	6.47	6.01	5.36	4.53	3.58	2.56	1.42	0.22	...
" 29	6.66	6.29	5.74	5.00	4.08	3.09	2.00	0.80	...	...
Nov. 13	6.56	6.12	5.48	4.72	3.75	2.66	1.48	0.25	...	...
" 28	6.46	5.95	5.26	4.42	3.36	2.28	1.08	...	...	...
Dec. 13	6.40	5.86	5.13	4.25	3.28	2.06	0.87	...	...	...

A close agreement, however, could not reasonably be expected; for the Intensities represent the sun's effect at the summit of the atmosphere, but the Temperatures, at its base. Indeed, the sun's intensity upon the exterior of the earth's atmosphere, like the fall of rain or snow, is a primary and distinct phenomenon. While passing through the atmosphere to the earth, the solar rays are subject to refraction, absorption, polarization and radiation; also to the effects of evaporation, of winds, clouds, and storms. Thus the heat which finally elevates the mercurial column of the Thermometer, is the resultant of a variety of causes, a single thread in the network of solar and terrestrial phenomena.

There is still a general agreement of the delineated curves of intensity with actual phenomena. Should the inquiry be made, in what part of the earth the sun's intensity continues most uniform for the longest period, an inspection of the flexures of the curves (Plate I), at once indicates the region intermediate between the Equator and the Tropic of Cancer, on the one side, and of Capricorn on the other.[1] Thus the curve for latitude 10° shows the solar intensity to be nearly stationary during half the year, from March to September. During October and November, it falls rapidly, and after remaining nearly unchanged for a few days in December, it again rises rapidly in January and February. As the sun's heat is the prime cause of winds, we might infer that this region would be comparatively calm during the half year mentioned, and that in the remaining months there would be greater atmospheric fluctuations.

Such were the general indications of Plate I, representing the *amounts;* and, on recurring to Plate II, representing the *rates* of diurnal intensity, the status is precisely similar, except that the region of summer calm is removed further from the equator, and nearer to the tropic. On referring to a recent work on the Physical Geography of the Sea, with respect to this circumstance, I find that "the variables," or calms of Cancer and of Capricorn, occur in the very latitudes thus indicated by the compound effect of the amount and rate of solar intensity. And further, the annual range of solar intensity, which is least upon the equator, has its counterpart in the belt of equatorial calms, or "doldrums." The same effect extends also to the ocean itself, and appears in the tranquillity of the Sargosso Sea. While the curves of intensity for the higher latitudes are significant hieroglyphs of the serenity of summer, and the more violent winds and storms of March and September. The entire deprivation of the sun's intensity during a part of the year, within the Arctic and Antarctic circles, may also produce a Polar calm, at least during the depth of winter. But the existence of such calm, though probable, can neither be disproved nor verified, as the pole appears not to have been approached nearer than within about five hundred miles. Parry and Barrow believed that a perfect calm exists at the Pole.

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[1] The connection of the curves of the Sun's Intensity with the lines of Equatorial and Tropical calms, was suggested by Prof. Henry.

SECTION V.

FORMULA AND TABLE OF THE SUN'S ANNUAL INTENSITY UPON ANY LATITUDE OF THE EARTH.

By the method explained in the last Section, the diurnal intensity, in a vertical direction, might be computed for each and every day in the year, and the sum total would evidently represent the Annual Intensity.

The sum of the daily intensities for a month, or monthly intensities, might be found in the same manner. But, instead of this slow process, we shall first find an analytic expression for the aggregate intensity during any assigned portion of the year, and then for the whole year. The summation is effected by an admirable theorem, first given by Euler; a new investigation of which, with full examples by the writer, may be found in the *Astronomical Journal* (Cambridge, Mass.), Vol. II, p. 121. Thus, let u denote the xth term of a series, where u is a function of x. Attributing to x the successive values 1, 2, 3, 4, ... x, and denoting the sum of the results by $\Sigma\, u$, it is shown that,

$$\Sigma u = \int u dx + \tfrac{1}{2}\, u + \tfrac{1}{12}\frac{du}{dx} - \tfrac{1}{720}\frac{d^3u}{dx^3} + \tfrac{1}{30240}\frac{d^5u}{dx} - \ldots + C. \qquad (20.)$$

Since this important formula has not yet been introduced into any American treatise on the Calculus, I here insert one of the two demonstrations from the Journal referred to, which indeed was suggested by the present research:—

Imagine the several terms of the original series to be ordinates of a curve, and erected at a unit's distance from each other, along an "axis of X;" then, by the well-known formula of the Calculus, $\int u\, d\, x$ will represent the area of this curve.

Again, connecting the upper adjacent extremities of the ordinates by straight lines, there will be represented an inscribed semi-polygon made up of parallel trapezoids whose bases are each equal to unity, and their areas equal to $\frac{1}{2}\,(0 + F_{(1)}) + \frac{1}{2}\,(F_{(1)} + F_{(2)}) + \ldots \frac{1}{2}\,(F_{(x-1)} + F_{(x)})$; adding the contiguous half terms, it becomes $\Sigma\, F_{(x)} - \frac{1}{2}\, F_{(x)}$, or $\Sigma\, u - \frac{1}{2}\, u$.

Between each trapezoid and the curved line above it, is a small segment; and if $f(x)$ or u' denote the area of the last or xth segment, then $\Sigma\, f\,(x)$ or $\Sigma\, u'$ will denote their collective area. The whole curve being made up of the inscribed semi-polygon and these segments, we have

$$\int u\, d\, x = \Sigma\, u - \tfrac{1}{2}\, u + \Sigma\, u',$$

or

$$\Sigma\, u = \int u\, d\, x + \tfrac{1}{2}\, u - \Sigma\, u'.$$

With respect to the last term, suppose u' to be referred to a new curve, as has already been done for u, and so on; then,

$$\Sigma\, u' = \int u'\, d\, x + \tfrac{1}{2}\, u' - \Sigma\, u'',$$

$$\Sigma\, u'' = \int u''\, d\, x + \tfrac{1}{2}\, u'' - \Sigma\, u''', \ldots$$

Subtracting the last of these three equations from the preceding, and that result from the first, and cancelling,

$$\left.\begin{aligned}\Sigma u = \int u\,dx + \tfrac{1}{2}u - \int u'\,dx - \tfrac{1}{2}u' \\ + \int u''\,dx + \tfrac{1}{2}u'' \\ - \ldots\ldots - \ldots\ldots\end{aligned}\right\}$$

It is now necessary to determine u' in terms of u, or of x. Recurring to the last segment of the curve above referred to, it is evident that its area above the trapezoid, and denoted by u', is equal to

$$u' = \int F_{(x)}\,dx - \int F_{(x-1)}\,dx - \tfrac{1}{2}(F_{(x)} + F_{(x-1)}).$$

Developing by Taylor's theorem; since $u = F_{(x)}$,

$$F_{(x-1)} = F_{(x)} - \frac{du}{dx} + \frac{d^2u}{1.2\,dx^2} - \frac{d^3u}{1.2.3\,dx^3} + \ldots.$$

$$\int F_{(x-1)}\,dx = \int F_{(x)}\,dx - u + \frac{du}{1.2\,dx} - \frac{d^2u}{1.2.3\,dx^2} + \ldots.$$

Substituting the two right-hand values in the former equation, the first terms will cancel each other, leaving

$$u' = -\tfrac{1}{12}\frac{d^2u}{dx^2} + \tfrac{1}{24}\frac{d^3u}{dx^3} - \tfrac{1}{80}\frac{d^4u}{dx^4} + \ldots.$$

That is, each derived function is equal to $-\frac{1}{12}$th of the second differential coefficient of the preceding, $+\frac{1}{24}$th of the third, &c.

$$u'' = \tfrac{1}{144}\frac{d^4u}{dx^4} - \ldots.$$

$$-\tfrac{1}{2}u' + \tfrac{1}{2}u'' - \ldots\ldots = \tfrac{1}{24}\frac{d^2u}{dx^2} - \tfrac{1}{48}\frac{d^3u}{dx^3} + \tfrac{7}{720}\frac{d^4u}{dx^4} - \ldots.$$

$$-\int u'\,dx + \int u''\,dx - \ldots\ldots = \tfrac{1}{12}\frac{du}{dx} - \tfrac{1}{24}\frac{d^2u}{dx^2} + \tfrac{7}{360}\frac{d^3u}{dx^3} - \ldots.$$

Substituting these last two values in the equation above,

$$\Sigma u = \int u\,dx + \tfrac{1}{2}u + \tfrac{1}{12}\frac{du}{dx} - \tfrac{1}{720}\frac{d^3u}{dx^3} \ldots. + C,$$

as was to be demonstrated. Let it now be applied to different examples of series, whose xth term is a function of x.

I. To find the sum of the *arithmetical progression*,

$$d + 2d + 3d + \ldots. + xd = \Sigma u.$$

Here $u = xd$; $\int u\,dx = \frac{1}{2}x^2 d$; $\frac{du}{dx} = d$.

Whence $\Sigma u = \frac{1}{2}x^2 d + \frac{1}{2}xd + \frac{1}{12}d + C$.

If $x = 1, d = \frac{1}{2}d + \frac{1}{2}d + \frac{1}{12}d + C$.

Subtracting, $\Sigma u = \frac{1}{2}x(xd + d)$; which result coincides with the common arithmetical rule.

II. To find the sum of the *geometrical progression*,

$$ar + ar^2 + ar^3 + \ldots. + ar^x.$$

$$\text{Here} \quad u = a\,r^x; \int u\,d\,x = \frac{a\,r^x}{\log r}.$$

$$\frac{d\,u}{d\,x} = a\,r^x \log r; \frac{d^3 u}{d\,x^3} = a\,r^x \log^3 r; \text{ \&c.}$$

The sum of the coefficients of $a\,r^x$ being constant, let it be denoted by B; then will

$$\Sigma\,u = B\,a\,r^x + C.$$

$$\text{If } x = 0, \quad a\,r = B\,a\,r + C.$$

$$\text{If } x = 1, \quad 0 = B\,a + C.$$

Whence $\Sigma\,u = \dfrac{a\,r^{x+1} - ar}{r - 1}$; which also agrees with the well known rule.

III. To find the sum of the *trigonometric series*,

$$\sin a + \sin 2\,a + \sin 3\,a + \ldots + \sin x\,a.$$

$$\text{Here} \quad u = \sin x\,a; \int u\,d\,x = -\frac{1}{a}\cos x\,a.$$

$$\frac{d\,u}{d\,x} = a\cos x\,a; \frac{d^3 u}{d\,x^3} = -a^3 \cos x\,a;$$

proceeding, therefore, as in II., we have

$$\Sigma\,u = \tfrac{1}{2}\sin x\,a + B\cos x\,a + C.$$

$$\text{If } x = 0, \quad 0 = B + C.$$

$$\therefore \Sigma\,u = \tfrac{1}{2}\sin x\,a + B(\cos x\,a - 1).$$

$$\text{If } x = 1, \sin a = \tfrac{1}{2}\sin a + B(\cos a - 1),$$

$$\text{And} \qquad B = \tfrac{1}{2}\frac{\sin a}{\cos a - 1} = -\tfrac{1}{2}\frac{\cos \frac{1}{2}a}{\sin \frac{1}{2}a}.$$

$$\Sigma\,u = \tfrac{1}{2}\sin x\,a - \frac{\cos \frac{1}{2}a(\cos x\,a - 1)}{2\sin \frac{1}{2}a}.$$

Reducing to a common denominator, we have by Trigonometry,

$$\Sigma\,u = \frac{\cos \frac{1}{2}a - \cos(x + \frac{1}{2})\,a}{2\sin \frac{1}{2}a} = \frac{\sin(x+1)\frac{1}{2}a \sin \frac{1}{2}x\,a}{\sin \frac{1}{2}a}.$$

The formula of summation has its failing cases; but these may be pointed out as plainly as those of Taylor's Theorem. Without entering here into a full discussion, it must apply in all cases where the summation is in its nature possible, and the differential co-efficients do not become infinite. It applies rigorously where the terms are all positive, and the differential co-efficient becomes zero, as in Example I.; also where the collective co-efficient can be represented by *a second constant*, denoted by B, and so can be eliminated, as in Example II. and III. Had not advantage been taken of this feature in the last Example, the sum were represented by the following series, which still converges rapidly when a does not much exceed unity:

$$\Sigma\,u, \text{ or } \Sigma \sin x\,a = -\frac{1}{a}\cos x\,a + \tfrac{1}{2}\sin x\,a + \tfrac{1}{12}a\cos x\,a - \tfrac{1}{720}a^3\cos x\,a + \ldots + C.$$

Having now demonstrated the formula of summation, let it be applied to (13) where the diurnal intensity is measured by

$$u = \Delta^2(\sin L \sin D.H + \cos L \cos D \sin H).$$

It may be remarked that the arc H can be developed in terms of its cosine; Δ^2 may be expressed in powers of $\cos\theta$; and thus u may be represented entirely in terms of the true longitude T; and ultimately in terms of the mean longitude or anomaly; as, $u = A + B\sin(b + a x) + C\sin(c + 2 a x) + D\sin(d + 3 a x) + \ldots$; where a or n denotes the Sun's daily motion in longitude, or arc 59′8″; which is .0172. This arc being so much less than unity, shows that the regular process of summation without a second constant, will converge with extreme rapidity, stopping at the first differential co-efficient, and leaves us at liberty to determine the sum $\int u\,dx + \frac{1}{2}u + \frac{1}{12}\frac{du}{dx}$ in such manner as may be most convenient.

Therefore, let x or t denote the number of days elapsed after the beginning of the year or epoch; n being the mean daily motion in longitude; $a' + n t$ or $a' + n x$, the mean anomaly; T, the true longitude, and P the longitude of the perihelion, so that the true anomaly $\theta = T - P$, and $d\theta = dT$. Also, if ω denote the obliquity of the ecliptic, then by Astronomy, $\sin D = \sin\omega \sin T$.

Since $\cos H = -\tan L \tan D$, we have $\sin^2 H = 1 - \tan^2 L \tan^2 D$, or again $\cos^2 L \cos^2 D \sin^2 H = \cos^2 L \cos^2 D - \sin^2 L \sin^2 D$. Substituting in the last member, $1 - \sin^2 D$ for $\cos^2 D$, also 1 for $\cos^2 L + \sin^2 L$; then dividing by $\cos^2 L$, and taking the square root,

$$\cos D \sin H = \sqrt{1 - \frac{\sin^2 D}{\cos^2 L}} = \sqrt{1 - \left(\frac{\sin\omega}{\cos L}\right)^2 \sin^2 T}. \qquad (21.)$$

With respect to Δ^2, let us here write its values from equation (8), and another value given by the ordinary polar equation of the ellipse; assuming A to be 1; and c, a new constant such that, since $d\theta$ is equal to dT,

$$\Delta^2 = \frac{c}{\rho^2} = \frac{c\,dT}{n\,dx\sqrt{1-e^2}} = \frac{c(1 + e\cos\theta)^2}{(1-e^2)^2}. \qquad (22.)$$

Substituting now the third members of the last two equations in place of the first members which occur in the preceding expression for u, and multiplying by dx,

$$u\,dx = \frac{c\,dT}{n\sqrt{1-e^2}} \left\{ \sin L \sin\omega \sin T . H + \cos L \sqrt{1 - \left(\frac{\sin\omega}{\cos L}\right)^2 \sin^2 T} \right\}. \qquad (23.)$$

The next step is to integrate this equation, where in the first term, $\sin\omega \sin T$ has been substituted for its equal, $\sin D$. The integral of the last term is readily identified as the arc of an ellipse whose eccentricity is $\frac{\sin\omega}{\cos L}$; therefore let

$$\int dT \sqrt{1 - \left(\frac{\sin\omega}{\cos L}\right)^2 \sin^2 T} = E, \textit{ an elliptic function of the second species.}$$

Again, integrating the variable factors of the first term by parts, $\int \sin T . H\,dT = -H\cos T + \int \cos T\,dH$. To obtain dH in a function of T, let us differentiate $\sin D = \sin\omega \sin T$, and $\cos H = -\tan L \tan D$, giving $\cos D\,dD = \sin\omega \cos T\,dT$, and $\sin H\,dH = \frac{\tan L\,dD}{\cos^2 D}$. Whence $dH = \frac{\tan L \sin\omega \cos T\,dT}{\sin H \cos D . \cos^2 D}$; or substituting for $\sin H \cos D$ its equal from (21), and for $\cos^2 D$ its equal $1 - \sin^2 \omega \sin^2 T$; then multiplying by $\cos T$, we have,

$$\int \cos T\, d\, H = \int \frac{\tan L \sin \omega \cos^2 T\, d\, T}{(1 - \sin^2 \omega \sin^2 T) \sqrt{1 - \left(\frac{\sin \omega}{\cos L}\right)^2 \sin^2 T}}.$$

Here in a changed form, $\sin \omega \cos^2 T = \sin \omega - \sin \omega \sin^2 T$, which is equal to $\frac{1}{\sin \omega} [(1 - \sin^2 \omega \sin^2 T) + \sin^2 \omega - 1]$; therefore writing $-\cos^2 \omega$ in place of $\sin^2 \omega - 1$, and then separating the expression into two parts, we obtain after cancelling the common factor,

$$\int \cos T d H = \int \frac{\tan L}{\sin \omega} \left\{ \frac{d\, T}{\sqrt{1 - \left(\frac{\sin \omega}{\cos L}\right)^2 \sin^2 T}} - \frac{\cos^2 \omega\, d\, T}{(1 - \sin^2 \omega \sin^2 T) \sqrt{1 - \left(\frac{\sin \omega}{\cos L}\right)^2 \sin^2 T}} \right\}. \quad (24.)$$

Now let $\int \frac{d\, T}{\sqrt{1 - \left(\frac{\sin \omega}{\cos L}\right)^2 \sin^2 T}} = F$, *an elliptic function of the first species;* and

$$\int \frac{d\, T}{(1 - \sin^2 \omega \sin^2 T) \sqrt{1 - \left(\frac{\sin \omega}{\cos L}\right)^2 \sin^2 T}} = \Pi,$$ *an elliptic function of the third species,* according to Legendre and other geometers.

Adopting these designations, we have now defined the terms of $\int u\, d\, x$. Passing over $\frac{1}{2}\, u$, as already known, the next term of the general formula of summation (20), is $\frac{1}{12} \frac{d\, u}{d\, x}$; which is determined as follows: Taking the logarithmic differential of (13) in its simplified form,

$$u = \Delta^2 \sin L \sin \omega \sin T (H - \tan H).$$

$$\frac{d\, u}{u} = \frac{2 d \Delta}{\Delta} + \cot T\, d\, T + \left(1 - \frac{1}{\cos^2 H}\right) \frac{d\, H}{H - \tan H}.$$

Again, taking the logarithmic differential of the first and last members of (22), recollecting that $d\, \theta = d\, T$, we find $\frac{2 d \Delta}{\Delta} = \frac{-2\, e \sin \theta\, d\, T}{1 + e \cos \theta}$. Also equating the first and third members of (22), $\frac{1}{d\, x} = \frac{\Delta^2\, n \sqrt{1 - e^2}}{c\, d\, T}$. And the value of $d\, H$ has already been found; whence by making the indicated substitutions and changes,

$$\frac{1}{12} \frac{d\, u}{d\, x} = \frac{1}{12} \frac{u\, \Delta^2\, n \sqrt{1 - e^2}}{c} \left\{ \frac{-2\, e \sin \theta}{1 + e \cos \theta} + \cot T - \frac{\tan^2 H \tan L \sin \omega \cos T}{(H - \tan H) \cos^3 D \sin H} \right\}.$$

The last term may be further simplified by multiplying and dividing by $\sin T$, then substituting $\sin D$ for $\sin \omega \sin T$, and $-\cos H$ for $\tan L \tan D$, and so cancelling $\tan H$, as shown in the result which follows. Referring to (20), and collecting the terms of summation represented by $\Sigma\, u$, we obtain the annexed general expression of the Sun's intensity for any assigned part of the year; thus,

$$\left.\begin{aligned}
&\int u\,dx = \frac{c.\cos L}{n\sqrt{1-e^2}}\left\{E - \tan L \sin\omega \cos T . H + \tan^2 L . F - \tan^2 L \cos^2\omega . \Pi\right\} \\
&+ \tfrac{1}{2}\,u = \tfrac{1}{2}\,\Delta^2 (\sin L \sin D . H + \cos L \cos D \sin H) \\
&+ \tfrac{1}{12}\frac{du}{dx} = \tfrac{1}{12} . \frac{u\,\Delta^2 n\sqrt{1-e^2}}{c}\left(\frac{-2\,e \sin\theta}{1+e\cos\theta} + \cot T + \frac{\tan H \cot T}{(H - \tan H)\cos^2 D}\right) \\
&\ldots + C = \ldots + C.
\end{aligned}\right\} \quad (25.)$$

Having thus obtained $\Sigma\, u$, we may regard it as an implicit function, varying continuously with the longitude T, which returns to the same value at the end of a tropical year. Taking, then, the sum of the above terms as an integral between the limits, $T = 360°$, and $T = 0$; the purely trigonometric terms and constant having the same values at the beginning and end of the year, will vanish, leaving only the three elliptic functions, multiplied as follows:—

$$\Sigma\, u' = \frac{c \cos L}{n\sqrt{1-e^2}}\left\{E'' + \tan^2 L . F'' - \tan^2 L \cos^2\omega . \Pi''\right\}. \qquad (26.)$$

Here the eccentricity or common modulus is $\frac{\sin\omega}{\cos L}$.

The Sun's Annual Intensity upon any latitude of the Earth is thus proportional to the sum of two Elliptic circumferences of the first and the second order, diminished by an Elliptic circumference of the third order.

On the Equator, L and $\tan L$ are 0, $\cos L$ is 1, and the expression reduces to $\Sigma\, u' = \frac{c\,E''}{n\sqrt{1-e^2}}$.... (27.) This proves that *the Sun's annual Intensity on the Equator is represented by the circumference of an ellipse, whose ratio of eccentricity is equal to the sine of the obliquity of the ecliptic.*

In the Frigid Zones, where the regular interchange of day and night in every twenty-four hours, is interrupted, the formula will require modification, though the general enunciation of the elliptic functions remains the same. The year in the Polar regions is naturally divided into four intervals, the first of which is the duration of constant night at mid-winter. The second interval at mid-summer is constant day; the third and fourth are intermediate spring and autumnal intervals, when the sun rises and sets in every twenty-four hours. For a criterion of the beginning and end of the winter interval, we evidently have $H = 0$; and for the limits of the summer interval $H = 12^h$.

During the winter interval, there is of course no solar intensity. The intensity of the spring and autumn intervals will be found by integrating (25) between the including limits, which results, added to that of the summer interval, give the annual intensity. First, then, to examine the summer interval; H is 12 hours or π, $\sin H$ is 0, and consequently by (23),

$\int u\,dx = \int \frac{c\,d\,T}{n\sqrt{1-e^2}} \sin L \sin\omega \sin T . \pi = -\frac{c \sin L \sin\omega \cos T . \pi}{n\sqrt{1-e^2}}$, which is precisely equal to the second term of (25), at the end of the spring, and at the beginning of the autumnal interval; so that on integrating between these limits, it will entirely disappear; and the same will apply to $\tfrac{1}{2}\,u + \tfrac{1}{12}\frac{du}{dx}$. For at the beginning of the spring, and end of the autumn interval, when H is 0, $\frac{1}{2} u$ becomes 0; and u being a zero factor, $\frac{1}{12}\frac{du}{dx}$ in (25) reduces to 0. Then exclusive of the three elliptic functions, the intensity of the spring interval will be $\frac{1}{2} u + \frac{1}{12}\frac{du}{dx} - 0$; that of the autumnal interval, $0 - \frac{1}{2} u' - \frac{1}{12}\frac{du'}{dx}$; and for the summer interval, $\frac{1}{2} u' + \frac{1}{12}\frac{du'}{dx} - \frac{1}{2} u - \frac{1}{12}\frac{du}{dx}$; the sum of which is evidently 0.

The expression of annual intensity thus reduces to the three elliptic functions in (26) integrated between the limits of the spring and autumnal intervals. Their collective differential in (23) and the analysis subjoined to it, will give, by making $\sin Z = \frac{\sin \omega \sin T}{\cos L}$,

$$u\,d\,x = \frac{c\,d\,T}{n\sqrt{1-e^2}}\left\{\cos L \cos Z + \frac{\sin L \tan L}{\cos Z} - \frac{\sin L \tan L \cos^2 \omega}{(1-\cos^2 L \sin^2 Z)\cos Z}\right\}.$$

Here $\cos L \cos Z + \frac{\sin L \tan L}{\cos Z}$ may take the form $\frac{\cos^2 L \cos^2 Z + \sin^2 L}{\cos L \cos Z}$, or, $\frac{1-\cos^2 L \sin^2 Z}{\cos L \cos Z}$, or $\frac{\sin^2 \omega}{\cos L \cos Z}\left[\frac{1}{\sin^2 \omega} - 1 + \left(1 - \frac{\cos^2 L}{\sin^2 \omega}\sin^2 Z\right)\right]$.

Again, differentiating the above value of $\sin Z$,

$$d\,T = \frac{\cos L}{\sin \omega}\cdot\frac{\cos Z}{\cos T}\cdot d\,Z = \frac{\cos L}{\sin \omega}\cdot\frac{\cos Z\,d\,Z}{\sqrt{1-\left(\frac{\cos L}{\sin \omega}\right)^2 \sin^2 Z}};\ \text{whence,}$$

$$u\,d\,x = \frac{c\,d\,Z}{n \sin \omega \sqrt{-e^2}}\left\{\frac{\cos^2 \omega}{\sqrt{1-\left(\frac{\cos L}{\sin \omega}\right)^2 \sin^2 Z}} + \sin^2 \omega \sqrt{1-\left(\frac{\cos L}{\sin \omega}\right)^2 \sin^2 Z} - \frac{\sin^2 L \cos^2 \omega}{(1-\cos^2 L \sin^2 Z)\sqrt{1-\left(\frac{\cos L}{\sin \omega}\right)^2 \sin^2 Z}}\right\};\ \text{or,}$$

$$\int u\,d\,x = \frac{c\,.\cos^2 \omega}{n \sin \omega \sqrt{1-e^2}}\left\{F + \tan^2 \omega\,.\,E - \sin^2 L\,.\,\Pi\right\}.$$

As before remarked, these three integrals are to be taken between the limits of the spring and autumnal intervals. At the beginning of the former and end of the latter, H is 0; whence $\cos H$ or $1 = -\tan L \tan D$; and $D = 90° - L$ taken with an opposite sign for south Declination. In this case,

$$\sin Z = \frac{\sin \omega \sin T}{\cos L} = -\frac{\sin D}{\cos L} = -1,\ \text{or}\ Z = 270°.$$

At the end of the spring, and at the beginning of the autumnal interval H is 12^h; $\cos H$ or $-1 = -\tan L \tan D$; whence D is $90° - L$, $\sin Z = 1$, or $Z = 90°$. Now the elliptic functions integrated between the limits, $Z = 270°$, $Z = 90°$ give semi-circumferences for the spring interval, and the same for the autumnal interval, the sum of which will be entire circumferences. We have, therefore, for the Annual Intensity in the Frigid Zones,

$$\Sigma\, u' = \frac{4\, c\, \cos^2 \omega}{n\, \sin \omega\, \sqrt{1 - e^2}} \left\{ F' + \tan^2 \omega \,.\, E' - \sin^2 L \,.\, \Pi' \right\}. \qquad (28.)$$

Here the eccentricity or common modulus of the three elliptic integrals is $\frac{\cos L}{\sin \omega}$, being the reciprocal of the modulus in (26); but the intensity is still denoted by three entire elliptic circumferences.

At the Poles, where L is 90°, and $\cos L$ is 0, the expression of annual intensity reduces to $\frac{2\, c\, \pi \sin \omega}{n \sqrt{1 - e^2}}$. (29.)

The three species of elliptic functions are known to represent four equal and similar quadrants, as in the ellipse. Extensive tables have been published by Legendre, of the numeric values of E and F; and in his *Traité des Fonctions Elliptiques,* Vol. I. p. 141, the value of the quadrant Π' is given in terms of E and F. Thus, if x denote an axillary arc, such that $c^2 \sin^2 x = n$, a negative quantity, less than 1; then,

$$\text{“}\Pi = \int \frac{d\,\phi}{(1 + n \sin^2 \phi)\sqrt{1 - c^2 \sin^2 \phi}}\,;\; \Pi' = F' + \frac{\tan x}{\sqrt{1 - c^2 \sin^2 x}} (F' E_{(x)} - E' F_{(x)}).\text{”}$$

Comparing with (24), $\sin^2 x = \cos^2 L$; $\Pi' = F' + \frac{\cot L}{\cos \omega} (F' E_{(x)} - E' F_{(x)})$; substituting this value into (26), we find for the *Annual Intensity in the Torrid and Temperate Zones,*

$$\Sigma\, u' = \frac{4\, c}{n \sqrt{1 - e^2}} \left\{ \begin{array}{l} [E' (\sin L \cos \omega \,.\, F_{(90^\circ - L)} + \cos L) + \\ F' . \sin L (\sin^2 \omega \tan L - \cos \omega \,.\, E_{(90^\circ - L)})] \end{array} \right\}. \qquad (30.)$$

We have heretofore denoted whole circumferences by the double accent, thus $E'' = 4\, E'$. In (30) E', and F' denote quadrants; $F_{(90^\circ - L)}$ and $E_{(90^\circ - L)}$ elliptic functions whose amplitude in Legendre's system is $90^\circ - L$; $\frac{\sin \omega}{\cos L}$ is the common modulus; L denoting the latitude of the place, and ω the obliquity of the ecliptic. The interpolation of Legendre's tables for second differences, is described in Vol. II. p. 202 of the *Fonctions Elliptiques.* From the Polar Circle to the Pole, $\frac{\cos L}{\sin \omega}$ will denote the common modulus, x becomes ω, and $\sqrt{1 - c^2 \sin^2 x} = \sin L$; hence by (28), *the Annual Intensity in the Frigid Zone is,*

$$\Sigma\, u' = \frac{4\, c}{n \sqrt{1 - e^2}} \left\{ \begin{array}{l} [E' (\sin L \cos \omega \,.\, F_{(\omega)} + \sin \omega) + \\ F' . \cos \omega \cos L (\cot \omega \cos L - \tan L \,.\, E_{(\omega)})] \end{array} \right\}. \qquad (31.)$$

With respect to the unit of measure for annual intensity, the mean tropical year contains 365.24 days; let this represent the annual number of vertical rays impinging on the equator; that is, *let the sun's intensity during a mean Equatorial day be taken as the thermal Unit,* and let the values for all the latitudes be converted in that proportion. Also denoting the annual intensity on the equator by 12, the mean equatorial *Month* may be used as another thermal unit. And taking the annual intensity on the equator as 81.5 *Units,* with reference to Brewster's formula, the intensity on other latitudes may be expressed in that proportion.

With the aid of Legendre's elliptical tables, and formulas (27), (30), (31), (29), the computation of annual intensities is entirely practicable. The results converted into units, with differences for every five degrees of latitude, have been carefully verified and tabulated as follows:—

The Sun's Annual Intensity.

Latitude.	Thermal units.	Thermal months.	Thermal days.	Diff. days.	Latitude.	Thermal units.	Thermal months.	Thermal days.	Diff. days.
0°	81.50	12.00	365.24	1.27	50°	55.73	8.21	249.74	20.92
5	81.22	11.96	363.97	3.78	55	51.06	7.52	228.82	21.06
10	80.38	11.83	360.19	6.28	60	46.36	6.83	207.76	19.91
15	78.97	11.63	353.91	8.70	65	41.92	6.17	187.85	14.81
20	77.03	11.34	345.21	11.01	70	38.61	5.69	173.04	9.82
25	74.57	10.98	334.20	13.20	75	36.42	5.36	163.22	6.59
30	71.63	10.55	321.00	15.30	80	34.95	5.15	156.63	3.80
35	68.21	10.04	305.70	17.15	85	34.10	5.02	152.83	1.24
40	64.39	9.48	288.55	18.76	90	33.83	4.98	151.59	0.00
45	60.20	8.86	269.79	20.05					

From this table it will be seen that, at the Tropic of Capricorn, or of Cancer, the Sun's annual Intensity is but eleven thermal months, being twelve on the Equator. In the latitude of New Orleans, the annual intensity in a vertical direction is ten and a half thermal months, and in the latitude of Philadelphia, nine and a half. At London the annual intensity is reduced to eight thermal months; and at the Polar Circle, to six months, being just one-half the value on the Equator. Thus the intensity irregularly decreases, till it terminates at the South or North Pole, where the annual intensity is but five thermal months.

Again, it will be interesting to note the analogy which the differences for every five degrees of latitude, in the last column of the table, bear to the corresponding differences of *height in the atmosphere which limit the region of perpetual snow.* It has been observed that the different heights of perpetual frost "decrease very slowly as we recede from the equator, until we reach the limits of the torrid zone, when they decrease much more rapidly. The average difference for every five degrees of latitude in the temperate zone is 1,318 feet, while from the equator to 30°, the average is only 664 feet, and from 60° to 80°, it is only 891 feet—important meteorological phenomena depend on this fact." (*Olmsted's Natural Philosophy.*) The differences of computed annual intensity in the table vary in a manner precisely similar. While in the Temperate Zone, the decrease for every five degrees of latitude is from 13 to 21 thermal days, yet it averages only about 6 thermal days within the Tropics and beyond the Polar circles. The line of congelation evidently rises in summer, and falls in winter, between certain limits.

With reference to the connection between these annual Intensities and the observed annual Temperatures, the analogy of the Centigrade scale shows that units of intensity may be converted into degrees Fahrenheit, by a multiplier and constants; thus, $d = (u - i)\, y + x$. Since the values of the multiplier y, and constants i, x are not precisely known, a graphical construction will be employed; and it is plain that if computed intensities and observed temperatures both follow the same law of change, their delineated curves will be symmetrical.

Therefore, taking the latitudes for ordinates, and the Annual Intensities in the table for abscissas, we obtain the curve of Annual Intensity (Plate III.); and, in the same manner, the curve of Annual Temperature. It will be seen, no doubt with interest, that the curve of annual intensity is almost symmetrical with that of European temperatures, observed mostly on the western side of that continent. But the curve of American temperature based on the U. S. Army Observations for places on the eastern portion of the continent, diverges from the curve of intensity, and indicates a special cause depressing these temperatures below the normal standard due to their latitudes.

At Key West, on the southern border of Florida, the divergence commences, and on proceeding northwardly, continually increases in magnitude; that is, so far as reliable observations have been made along the expanding breadth of the North American continent.

It were natural to suppose that the annual temperature would be defined by the annual number of heating rays from the sun. Indeed on and near the tropical regions, the curves of annual temperature and solar intensity are symmetrical. But in the polar regions, the irregularity of the intervals of day and night, and of the seasons, and various proximate causes, introduce a discrepancy, which the principle of annual average does not obviate. The laws of solar intensity, however, have been determined; the laws of terrestrial temperature will require a special and apparently more difficult analysis.

It has been inferred that there are two poles of maximum cold about the latitude of 80° north, and in longitudes 95° E. and 100° W. The fewness of the observations, however, in that remote Hyperborean region, leave this question still open to investigation. The more recent "isothermal lines of mean annual temperature" published by Prof. Dove of Berlin, in 1852, indicate but one pole of cold, and that is very near the geographical Pole.

SECTION VI.

AVERAGE ANNUAL INTENSITY OF THE SUN UPON A PART OR THE WHOLE OF THE EARTH'S SURFACE.

HAVING determined the value of Σu representing the Sun's vertical intensity upon a single unit or point of the Earth's surface, let us next ascertain the average annual intensity upon a larger area, a zone, or the entire surface of the globe. After which, we shall glance at some of the climatic alternations which are most clearly made known and interpreted by the mechanism of the heavens.

Regarding the earth as a sphere whose radius is unity, $\cos L$ will be the radius, and $2\pi \cos L$ the circumference of the parallel of latitude L. It is evident that the intensity upon a single point multiplied by the circumference $2\pi \cos L$, will express the sum of the intensities received upon the whole parallel of latitude; consequently $\Sigma u \,.\, 2\pi \cos L \,.\, d L$ integrated between the limits of L and L' will denote the sum of the intensities upon the zone or surface between the latitudes L and L'. By Geometry, the surface of this zone is proved to be equal to $(\sin L - \sin L')\, 2\pi$. Therefore the sum of the annual intensities divided by the surface, will evidently give u, the *average annual intensity* of the Sun upon a unit of surface in that zone, as follows:—

$$u = \frac{\int_{L'}^{L} \Sigma u \,.\, \cos L\, d L}{\sin L - \sin L'}. \qquad (32.)$$

To find the value of this integral, Σu must first be developed in terms of $\cos L$. It is shown in (23), and in the analysis following that equation, that the annual intensity, exclusive of terms cancelled by the integration, is

$$\Sigma u = \frac{4c}{n\sqrt{1-e^2}}\left\{\cos L \,.\, E' + \frac{\sin^2 L \sin^2 \omega}{\cos L}\int_0^{\frac{\pi}{2}} \frac{\cos^2 T\, d T}{(1-\sin^2\omega \sin^2 T)\sqrt{1-\frac{\sin^2\omega}{\cos^2 L}\sin^2 T}}\right\}. \qquad (33.)$$

From the well known formula for the rectification of the ellipse, we have in the first place,

$$\cos L \,.\, E' = \frac{\pi}{2}\left(\cos L - \tfrac{1}{4}\frac{\sin^2\omega}{\cos L} - \tfrac{3}{64}\frac{\sin^4\omega}{\cos^3 L} - \tfrac{5}{256}\frac{\sin^6\omega}{\cos^5 L} - \tfrac{175}{(128)^2}\frac{\sin^8\omega}{\cos^7 L} - \ldots\right). \qquad (34.)$$

Next, to find the value of the last integral, let the radical of the denominator be first developed, and its terms multiplied into the other factors separately; then, preparatory to integration, let each numerator be divided by its denominator, as follows:—

$$\frac{\cos^2 T}{(1-\sin^2\omega \sin^2 T)\sqrt{1-\frac{\sin^2\omega}{\cos^2 L}\sin^2 T}} = \frac{1-\sin^2 T}{1-\sin^2\omega \sin^2 T}\left(1 + \tfrac{1}{2}\frac{\sin^2\omega}{\cos^2 L}\sin^2 T + \tfrac{3}{8}\frac{\sin^4\omega}{\cos^4 L}\sin^4 T + \ldots\right).$$

(a) (b) (c) ....

$$(a) = \frac{1-\sin^2 T}{1-\sin^2\omega \sin^2 T} = \frac{1}{\sin^2\omega} + \frac{1-\frac{1}{\sin^2\omega}}{1-\sin^2\omega \sin^2 T}$$

$$(b) = \frac{1}{2\cos^2 L} \times \frac{\sin^2\omega \sin^2 T \cos^2 T}{1-\sin^2\omega \sin^2 T} = \frac{1}{2\cos^2 L}(-\cos^2 T + (a)).$$

$$(c) = \frac{3}{8\cos^4 L} \times \frac{\sin^4\omega \sin^4 T \cos^2 T}{1-\sin^2\omega \sin^2 T} = \frac{3}{8\cos^4 L}(-\sin^2\omega \sin^2 T\cos^2 T - \cos^2 T + (a)).$$

$$(d) = \frac{5}{16\cos^6 L}(-\sin^4\omega \sin^4 T\cos^2 T - \sin^2\omega \sin^2 T\cos^2 T - \cos^2 T + (a)).$$

$$(e) = \frac{35}{128\cos^8 L}(-\sin^6\omega \sin^6 T\cos^2 T - \sin^4\omega \sin^4 T\cos^2 T - \sin^2\omega \sin^2 T\cos^2 T - \cos^2 T + (a)).$$

Multiplying now each term by $d T$, and integrating between the limits of $T = \frac{\pi}{2}$, and $T = 0$, we obtain the following results:—

$\int (a)\, d\,T = \frac{\pi}{2\,sin^2\omega} - \frac{1}{2}\,cot^2\omega \int_0^{\frac{\pi}{2}} \frac{d\,(2\,T)}{1 - sin^2\omega\, sin^2 T}$. Here substituting $\frac{1}{2} - \frac{1}{2}\,cos^2 T$ for its equal $sin^2 T$, the last term will take the known form of $\int \frac{d\,\theta}{p + q\,cos\,\theta}$, where θ represents $2\,T$; and by the Calculus its value between the proper limits, reduces to $\frac{\pi}{\sqrt{p^2 - q^2}}$ or $\frac{\pi}{cos\,\omega}$. Hence $\int (a)\,d\,T = \frac{\pi}{2}\left(\frac{1 - cos\,\omega}{sin^2\omega}\right)$.

Since $\int_0^{\frac{\pi}{2}} cos^2 T\,d\,T = \frac{\pi}{4}$, we have $\int (b)\,d\,T = \frac{\pi}{2}\left(-\frac{1}{2} + \frac{1 - cos\,\omega}{sin^2\omega}\right)\frac{1}{2\,cos^2 L}$.

Since $sin^2 T\,cos^2 T$ may take the form $cos^2 T - cos^4 T$; the formula of the Integral Calculus readily gives $\int (c)\,d\,T = \frac{\pi}{2}\left(\frac{-sin^2\omega}{8} - \frac{1}{2} + \frac{1 - cos\,\omega}{sin^2\omega}\right)\frac{3}{8\,cos^4 L}$.

In like manner $\int (d)\,d\,T = \frac{\pi}{2}\left(-\frac{1}{16}\,sin^4\omega - \frac{1}{8}\,sin^2\omega - \frac{1}{2} + \frac{1 - cos\,\omega}{sin^2\omega}\right)\frac{5}{16\,cos^6 L}$.

And $\int (e)\,d\,T = \frac{\pi}{2}\left(-\frac{5}{128}\,sin^6\omega - \frac{1}{16}\,sin^4\omega - \frac{1}{8}\,sin^2\omega - \frac{1}{2} + \frac{1 - cos\,\omega}{sin^2\omega}\right)\frac{35}{128\,cos^8 L}$.

The general formula (33) may be written—

$$\Sigma\,u = \frac{4\,c}{n\sqrt{1 - e^2}}\left\{cos\,L\,.\,E' + \frac{sin^2 L\,sin^2\omega}{cos\,L}\left(\int (a)\,d\,T + \int (b)\,d\,T + \ldots\right)\right\}.$$

Here $\frac{sin^2 L}{cos\,L}$ can take the form of $-\,cos\,L + \frac{1}{cos\,L}$; substituting this value, and multiplying it into the series of terms denoted by $\int (a)\,d\,T + \int (b)\,d\,T + \ldots$, and adding the products to the series (34) for $cos\,L\,.\,E'$, we at length obtain,

$$\Sigma\,u = \frac{2\,c\,\pi}{n\sqrt{1 - e^2}}\left\{cos\,\omega\,cos\,L + \frac{1 - cos\,\omega}{2\,cos\,L} + \frac{1 - cos\,\omega - \frac{1}{2}\,sin^2\omega}{8\,cos^3 L} + \frac{1 - cos\,\omega - \frac{1}{2}\,sin^2\omega - \frac{1}{8}\,sin^4\omega}{16\,cos^5 L}\right.$$
$$\left. + \frac{5\,(1 - cos\,\omega - \frac{1}{2}\,sin^2\omega - \frac{1}{8}\,sin^4\omega - \frac{1}{16}\,sin^6\omega)}{128\,cos^7 L} + \frac{7\,(N - \frac{5}{128}\,sin^8\omega)}{256\,cos^9 L} + \ldots\right\}. \quad (35.)$$

In the last term, N denotes the series within the parenthesis of the preceding numerator. Now, taking the obliquity of the ecliptic ω at 23° 28′, and representing the particular value of $\Sigma\,u'$ on the equator by 365.24 *thermal days* as in the last Section, the *multiplier* for converting other values into thermal days, will evidently be $\frac{365.24}{\Sigma\,u'}$, or $365.24 \div \frac{2\,c\,\pi}{n\sqrt{1 - e^2}}$ (.9590919); the latter being the value of $\Sigma\,u'$ when L is 0, and $cos\,L$ is 1. In this manner, and denoting the logarithms of the co-efficients by brackets, we find for the present century,

$$\left.\begin{array}{l}\Sigma\,u = [2.543225]\,cos\,L + [1.197235]\,sec\,L + [\bar{1}.211695]\,sec^3 L + \\ \quad [\bar{3}.819015]\,sec^5 L + [\bar{4}.616548]\,sec^7 L + [\bar{5}.509114]\,sec^9 L + \ldots\end{array}\right\}. \quad (36.)$$

Or in numbers,

$$\Sigma\,u = 349.322\,cos\,L + \frac{15.748}{cos\,L} + \frac{0.1628}{cos^3 L} + \frac{0.00659}{cos^5 L} + \frac{0.000414}{cos^7 L} + \ldots \quad (37.)$$

These formulas of Annual Intensity are applicable to the Torrid and Temperate Zones, and would have given those portions of the table in the last section with nearly the same facility as elliptic functions, but for the slow convergence of the series in the higher latitudes; the elliptic expressions are also preferred for the future case of secular values.

Denoting the co-efficients of (36) by $a, b, c, \ldots$ and with reference to formula (32); multiplying by $\cos L\, d\, L$, and integrating,

$$\Sigma\, u \,.\, \cos L\, d\, L = a \cos^2 L \,.\, d\, L + b \,.\, d\, L + \frac{c \,.\, d\, L}{\cos^2 L} + \frac{d \,.\, d\, L}{\cos^4 L} + \frac{e \,.\, d\, L}{\cos^6 L} + \ldots$$

$$\left.\begin{aligned} \int \Sigma\, u \,.\, \cos L\, d\, L &= a\left(\tfrac{1}{2} L + \tfrac{1}{2} \sin L \cos L\right) + b\, L + c \tan L + d\, \frac{\tan L}{3}\left(\frac{1}{\cos^2 L} + 2\right) \\ &+ e\left(\frac{\sin L}{5 \cos^5 L} + \frac{4}{5}\int \frac{d\, L}{\cos^4 L}\right) + \int\left(\frac{\sin L}{7 \cos^7 L} + \frac{6}{7}\int \frac{d\, L}{\cos^6 L}\right) + \ldots C \end{aligned}\right\} \quad (38.)$$

The last two integrals are given in the respective preceding terms. To determine the correction C, make L equal to 0; in this case, the surface being 0, the left hand member and all the other terms vanish, except C, which is, consequently, 0.

The next process is to find a similar formula for the Frigid Zone. Accordingly from (28), and the analysis preceding that equation, we have,

$$\Sigma\, u = \frac{4\, c}{n \sqrt{1 - e^2}} \left\{ \sin \omega . E' + \int_0^{\frac{\pi}{2}} \frac{\frac{\cos^2 \omega}{\sin \omega} . \cos^2 L \cos^2 Z\, d\, Z}{(1 - \cos^2 L \sin^2 Z) \sqrt{1 - \frac{\cos^2 L}{\sin^2 \omega} \sin^2 Z}} \right\} . \quad (39.)$$

This equation has precisely the form of (33); but ω there, corresponds to $90° - L$ here; and L there, corresponds to $90° - \omega$ here; T there, corresponds to Z here, and has the same limits of integration. Hence, by making the proper substitutions in (35) we may pass at once to the series for the annual intensity in the Frigid Zone, as here subjoined.

$$\begin{aligned} \Sigma\, u = \frac{2\, c\, \pi}{n \sqrt{1 - e^2}} &\left\{ \sin L \sin \omega + \frac{1 - \sin L}{2 \sin \omega} + \frac{1 - \sin L - \frac{1}{2} \cos^2 L}{8 \sin^3 \omega} + \frac{1 - \sin L - \frac{1}{2} \cos^2 L - \frac{1}{8} \cos^4 L}{16 \sin^5 \omega} \right. \\ &\left. + \frac{5 (1 - \sin L - \frac{1}{2} \cos^2 L - \frac{1}{8} \cos^4 L - \frac{1}{16} \cos^6 L)}{128 \sin^7 \omega} + \ldots \right\} . \end{aligned} \quad (40.)$$

Multiplying this equation by $\cos L\, d\, L$, or $D \sin L$, and integrating,

$$\left.\begin{aligned} \int \Sigma\, u \,.\, \cos L\, d\, L &= \frac{2\, c\, \pi}{n \sqrt{1 - e^2}} \left\{ \frac{\sin \omega \sin^2 L}{2} + \frac{\sin L - \frac{1}{2} \sin^2 L}{2 \sin \omega} \right. \\ &+ \frac{\sin L - \frac{1}{2} \sin^2 L - \frac{1}{8}\left(\frac{\sin 3L}{3} + 3 \sin L\right)}{8 \sin^3 \omega} + \frac{N - \frac{1}{28}\left(\frac{\sin 5L}{5} + \frac{5}{3} \sin 3L + 10 \sin L\right)}{16 \sin^5 \omega} + \ldots C \end{aligned}\right\} . \quad (41.)$$

Here N denotes the numerator of the preceding fraction. Now, integrating between the limits $L = 90°$, and $L = 90° - \omega = 66°\ 32'$, also introducing the constant multiplier described after (35), we shall find for the Frigid Zone,

$\int \Sigma\, u \,.\, \cos L\, d\, L = [2.580718] \left\{ \frac{\sin^3 \omega}{2} + \frac{\frac{1}{2} - \cos \omega + \frac{1}{2} \cos^2 \omega}{2 \sin \omega} + \ldots \right\}$, which is equal to 13.733.

Again, by formula (38), taking L between the limits 0 and $23°\ 28'$, we find the like sum between the equator and tropic, for the Torrid Zone, to be 141.86.

And taking L between the limits 23° 28′ and 66° 32′, the like sum for the Temperate Zone is 143.46

Substituting these values in equation (32), dividing by the denominator, and then converting into the same thermal measures, which were employed in the last Section, we obtain these final results:—

The Sun's Average Annual Intensity.

	Thermal days.	Thermal months.	Thermal units.
Upon the Polar Zones	166.04	5.45	37.05
" " Temperate Zones	276.38	9.08	61.67
" " Torrid Zone	356.24	11.70	79.49
" " whole Earth	299.05	9.83	66.73

Thus it appears that the Sun's annual intensity upon the whole earth's surface from pole to pole, averages 299 thermal days, being about five-sixths of the value on the equator.

Though the figures in the last column are strictly units of *intensity*, yet as shown by the curves, they also approximately represent annual *temperatures*, except near the Poles. Following these indications, the mean annual temperature of the whole earth's surface must be somewhat below 66° Fahrenheit. In comparison with this result, the mean annual temperature found by Prof. Dove, from a vast number of observations, may be introduced, which is approximately 58°.1 Fahrenheit. The like value found from the formula of Brewster, is $\int_0^{\frac{\pi}{2}} 81^\circ.5 \cos^2 L \, d \, L$, which is 64°. 0 Fahrenheit.

SECTION VII.

ON SECULAR CHANGES OF THE SUN'S INTENSITY.

In relation to secular variations of intensity, we shall adopt the hypothesis that the physical constitution of the sun has remained constant. The secular changes here considered, therefore, are those which depend solely on position and inclination, according to the laws of physical astronomy.

The recurrence of Spots on the Sun's disc, has lately been discovered to observe a regular periodicity. But their influence upon temperature appears to be insufficient for taking account of them.[1] A writer in the *Encyclopædia Britannica*, article Astronomy, states that "in 1823 the summer was cold and wet, the thermometer at Paris rose only to 23°.7 of Reaumur, and the sun exhibited no spots; whereas, in

[1] M. R. Wolf, in the *Comptes Rendus*, XXXV, p. 704, communicates his discovery that the minima of solar spots occur in regular periods of 11.111 years, or nine cycles in a century—and that the years in which the spots are most numerous are generally drier and more productive than the others—the latter being more humid and showery. Counsellor Schwabe, after twenty-six years of observation, does not think that the spots exert any influence on the annual temperature.

the summer of 1807, the heat was excessive, and the spots of vast magnitude. Warm summers, and winters of excessive rigor have happened in the presence or absence of the spots."[1]

Proceeding now to investigation, our first inquiry will relate to *changes of the sun's annual intensity upon the earth's surface regarded as one aggregate.* In Section II, formula (10), let the accented letters refer to the earth at an antecedent or future epoch; then, since Astronomy proves that the semi-transverse axis A is invariable, we have for the proportion of intensity at the secular epoch $\sqrt{\frac{1-e^2}{1-e'^2}}$. (42.)

In the *Connaissance des Tems*, for 1843, Leverrier has exhibited the secular values of most of the elements of the planetary orbits during 100,000 years before and after Jan. 1, 1800. The eccentricity of the earth's orbit at the present time being .0168, the value 100,000 years ago, and the greatest in that interval was .0473. Substituting these in the preceding expression, we find that the sun's annual intensity at the former epoch was greater than at present by one-thousandth part. Now this fraction of 365.24 days, counting the days at twelve hours each in respect to solar illumination, amounts to *between four and five hours of sunshine in a year*; and by so small a quantity only has the sun's annual intensity, during 100,000 years past, ever exceeded the yearly value at the present time. Nor can it depart from its present annual value by more than the equivalent of five hours of average sunshine in a year, for 100,000 years to come.

The superior and *ultimate limit* given by Leverrier, to which the eccentricity of the earth's orbit may have approached at some very remote but unknown period or periods, is .0777. At such epoch, the annual intensity is computed, as before, to have exceeded the intensity of the present by *thirteen hours* of sunshine in a year. On the other hand, the inferior limit of eccentricity being near to zero, indicates only *four minutes* of average sunshine in a year, less than the present annual amount. Between these two extreme limits, all annual variations of the solar intensity, whether past or future, must be included, even from the primitive antediluvian era, when the sun was placed in his present relation to the earth. By the third law of Kepler, on which equation (10) is based, these results are rigorous for siderial years; and by reason of the slight but nearly constant excess, the same may be concluded of tropical or civil years. For the annual variation of the tropical year is only —0d.000 000 066 86.

The preceding conclusions, it is proper again to observe, refer to the whole earth's surface collectively. Let us, in the next place, inquire concerning *changes of annual intensity upon the different Latitudes* of the earth. According to formulas (30) and (31), this variation will be a function of the eccentricity e, and the obliquity ω. For the present, let it be proposed to compute the annual intensity for an epoch 10,000

[1] Professor Henry was the first to show, by projecting on a screen in a dark room the image of the sun from a telescope with the eye glass drawn out, that the temperature of the spots was slightly less than that of the other parts of the solar disc. The temperature was indicated by a delicate thermo-electrical apparatus. Professor Sechi, of Italy, afterwards obtained the same result.—See *Silliman's Journal*, Vol. XLIX, p. 405.

years prior to A. D. 1800. The eccentricity of the orbit, e', was then .0187, according to Leverrier; and for the obliquity of the ecliptic, the most correct formula is probably that of Struve and Peters, quoted in the *American Nautical Almanac*. It is true, their formula may not strictly apply for so distant a period; but, since the value 24° 43′ falls within the maximum assigned by Laplace, it must be a compatible value, though its epoch may be somewhat nearer or more remote than 10,000 years. Therefore, substituting this value of ω, 24° 43′ in equations (30), (31) and multiplying by $\sqrt{\frac{1-e^2}{1-e'^2}}$, in order to substitute the proper eccentricity, and comparing the computed results with the table for 1850, given in Section V, as a standard, we find the annual intensity on the equator, at the former period, to have been 1.65 thermal days less than in 1850; the differences for every ten degrees of latitude are as follows:—

Change of the Sun's Annual Intensity 8,200 *Years B. C., from its Value in A. D.* 1850, *taken as the Standard.* (Plate III.)

Latitude	Difference in thermal days.	Latitude.	Difference in thermal days.	Latitude.	Difference in thermal days.
0°	—1.65	30°	—.96	60°	+2.11
10°	—1.58	40°	—.22	70°	+5.52
20°	—1.32	50°	+.68	80°	+7.18
				90°	+7.64

These results are exhibited graphically also on Plate III; from which it appears that the annual intensity within the Torrid Zone ten thousand years ago, averaged one thermal day and a half less than now; while from 35° of latitude to 50°, comprehending the whole area of the United States, it was virtually the same as at the present day. But above 50° of latitude, the annual intensity was then greater in an increasing rate towards the Pole, at which point it was between seven and eight thermal days greater than at the present time; in other words, the Poles both North and South, 10,000 years ago received twenty rays of solar heat in a year, where they now receive but nineteen. Owing to change in the obliquity of the ecliptic, the Sun may be compared to a swinging lamp; at the former period, it apparently moved farther to the north and to the south, passing more rapidly over the intermediate space.

The maximum variation of the obliquity of the ecliptic according to Laplace, without assigning its epoch, is 1° 22′ 34″, above or below the obliquity 23° 28′ in the year 1801.[1] Now the difference recognized in our calculation almost reaches this limit, being 1° 15′. As the secular perturbations are now understood, therefore, it follows that, since the Earth and Sun were placed in their present relation to each other, the annual intensity upon the Temperate zones has never varied (Plate III); between the Tropics, it has never departed from its present annual amount by more than about $\frac{1}{240}$th part, and is now very slightly increasing. The most perceptible

[1] Mécanique Céleste, Vol. II, p. 856, note, Bowditch's translation.

difference is in the Polar regions, where the secular change of annual intensity is more than four times greater than on the Equator; in its annual amount, the Polar cold is now very slowly increasing from century to century, which effect must continue so long as the obliquity of the ecliptic is diminishing. And thus, so far as relates to a decreased annual intensity, the celebrated "North-west passage" through the Arctic sea will be even more difficult in years to come than in the present age.

Having now considered the secular changes of annual intensity upon the earth and its different latitudes, let us next examine the *secular changes of intensity in relation to the Northern and Southern hemispheres.* The earth is now nearest the sun in winter of the northern hemisphere on January 1st, and farthest from the sun in summer, on July 4th. This collocation of times and distances has the advantage of rendering the extreme of summer cooler, and of winter, north of the equator, warmer than it would be at a mean distance from the sun. But south of the equator, on the contrary, it exaggerates the extremes by rendering the summer hotter and the winter colder. Before estimating this difference, we may observe that the perigee advances in longitude 11″.8 annually; by which the instant when the earth is nearest the sun, will date about five minutes in time later every year. The time of perihelion which now falls in January, will at length occur in February, and ultimately return to the southern hemisphere the advantage which we now possess. Indeed, it is remarkable that the perigee must have coincided with the autumnal equinox about 4,000 B. C., which is near the time that chronology assigns for the first residence of man upon the earth.

For ascertaining the difference of intensity, we know that the sun's declination goes through a nearly regular cycle of values in a year. The formula $\cos H = -\tan L \tan D$ then shows that the length of the day in the southern hemisphere is the same as in the northern hemisphere about six months earlier. Recurring to formula (18), it appears that the difference of intensities will then depend chiefly on the values of Δ^2. Now, for the northern winter on January 1st, Δ^2 is proportional to $\frac{1}{(1-e)^2}$; for winter in the southern hemisphere, July 4th, it is as $\frac{1}{(1+e)^2}$. The ratio of daily intensity of the northern, is to the southern then as one to $\left(\frac{1-e}{1+e}\right)^2$; or as 1 to $1-4e$ nearly; that is, 1 to $1-\frac{1}{15}$. And the like ratio for the summer intensities is as 1 to $1+\frac{1}{15}$. But $\frac{1}{15}$ is the extreme deviation for a few days only; the mean between this and 0, or $\frac{1}{30}$, would seem more correctly to apply to the whole seasons of summer and winter. Taking then $\frac{1}{30}$th of the greatest and least values of daily intensity, Section IV, for the temperate zone, it appears that winter in the southern hemisphere is now about 1° colder, and summer 3° hotter than in the northern hemisphere. The intensities during spring and autumn may be regarded as equal in both hemispheres. And the summer season of the south temperate zone being hotter, is also shorter by about eight days, owing to the rapid motion of the earth about the perihelion.

In confirmation of these last deductions, the younger Herschel refers to the glow and ardor of the sun's rays under a perfectly clear sky at noon, and observes, "one-fifteenth is too considerable a fraction of the whole intensity of sunshine, not to aggravate, in a serious degree, the sufferings of those who are exposed to it without shelter. The accounts of these sufferings in the interior of Australia, would seem far to exceed what have ever been experienced by travellers in the northern deserts of Africa. The author has observed the temperature of the surface soil in South Africa, as high as 159° Fahrenheit. The ground in Australia, according to Capt. Sturt, was almost a molten surface, and if a match accidentally fell upon it, it immediately ignited." (*Herschel's Astronomy.*)

The phenomenon is of sufficient interest to warrant a glance at the secular values. The eccentricity, 100,000 years ago, has already been stated at .0473; and the formula of the proportional general difference of the winter intensities, in the northern and southern hemispheres $1 - 2e$, becomes $1 - .0946$; and the maximum difference $1 - 4e$ becomes $1 - .1892$. Thus the difference of winter intensities between the northern and southern hemispheres, and likewise of summer intensities, was then about three times greater than at the present time. But this wide fluctuation of summer and winter intensities, in relation to the two hemispheres, scarcely affected the aggregate *annual* intensities, as before shown.

From occasional *Historic notices of climate*, it has been assumed that the winter season in Europe was formerly colder than at the present time. The rivers Rhine and Rhone were frozen so deep as to sustain loaded wagons; the Tiber was frozen over, and snow at one time lay forty days in the city of Rome; but the history of the weather presents winters of equal severity in modern times.[1] In the United States, likewise, since the period of our colonial history, the indications of an amelioration of climate are not conclusive. The great snow of February, 1717, rose above the lower doors of dwellings, and in the winters which closed the years 1641, 1697, 1740, and 1779, the rivers were frozen, and Boston and Chesapeake bays were at times covered with ice as far as the eye could reach; but the like occurs at similar intervals in our day. Mild winters, too, have intervened, and the other seasons are also very variable. The general indications, however, give rise to the question, whether there is a cause of change of climate in the course of the sun?

About two thousand years ago, in the time of Hipparchus, 128 B. C., the obliquity of the ecliptic, or the sun's greatest declination, was 23° 43′. It has now decreased to 23° 27½′; therefore, at the former epoch, the sun came farther north and rose to a higher altitude in summer; and went farther south and rose only to a lower altitude in midwinter. There is then an astronomic cause of change, of which we propose to determine more precisely the effect. For this purpose, the formula of daily intensity (18) may be written,

$$u = [1.90746] \left(\frac{1 + e \cos (T - P)}{1 - e^2}\right)^2 \sin L \sin D \, (\tan H \pm H).$$

[1] Thus, in the famous winter of 1709, thousands of families perished in their houses; the Arabic Sea was frozen over, and even the Mediterranean. The winter of 1740 was scarcely inferior, and snow lay ten feet deep in Spain and Portugal. In 1776 the Danube bore ice five feet deep below Vienna.

Here, for Δ, there is substituted its equal $\frac{1+e\cos(T-P)}{1-e^2}\,960''.9$; also generally $\sin D = \sin\omega \sin T$, and $\cos H = -\tan L \tan D$. For secular values, if t denote the number of years after, and $-t$ before, the year 1800,

$$e = 0.0167836 - .0000004163\,t;\ P = 279°31'10'' + 1'.0315\,t;$$

$$\text{Mean obliquity } \omega = 23°27'54'' - 0''.4645\,t - 0''.0000014\,t^2.$$

At the solstices of summer and winter T is 90° or 270°, and D is ω; also let the latitude L be 40°, which is nearly the latitude of Philadelphia, also of southern Italy and Greece. Computing now for B. C. 128, and for A. D. 1850, the daily intensities at the summer solstice are 90.45 and 90.05 thermal units, and at the winter solstice 28.67 and 29.04 respectively. The differences .40 and .37 must correspond almost precisely to degrees of the thermometer; and halving them for the whole seasons as before described, we are conducted to the following conclusion. In the time of Hipparchus, or about a century before Julius Cæsar, Virgil, Horace and Ovid flourished, *under the latitude of Italy and Greece the summer was two-tenths of a degree Fahrenheit hotter, and the winter as much colder, than at the present day.* The similar changes of solar intensity upon the United States in two hundred years, can only be made known by theory, and are evidently very slight. There has been, therefore, no sensible amelioration of climate in Europe or America from astronomical causes. The effect, however, of cutting down dense forests, of the drainage and cultivation of open grounds and woodlands admit of conflicting interpretation, and appear but secondary to the atmospheric fluctuations which are governed by the changes in the relative position of the earth and sun.

Before leaving the subject, the inquiry may arise respecting *Geological changes*, whether the secular inequalities have ever been of such value under the present order, as to admit of tropical plants growing in the temperate or frigid zones. In reply, as the annual intensity could never have varied in any considerable degree, the change must consist entirely in tempering the extremes of summer and winter to a perpetual spring. And this could not happen on both sides of the equator at once; for the same arrangement which made the daily intensities in the northern hemisphere equable, would subject those of the southern to violent alternations; and the wide breadth of the torrid zone would prevent the effects being conducted from one hemisphere to the other.

Let us then look back to that primeval epoch when the earth was in aphelion at midsummer, and the eccentricity at its maximum value—assigned by Leverrier near to .0777. Without entering into elaborate computation, it is easy to see that the extreme values of diurnal intensity, in Section IV, would be altered as by the multiplier $\left(\frac{1\pm e}{1\pm e'}\right)^2$, that is $1-0.11$ in summer, and $1+0.11$ in winter. This would diminish the midsummer intensity by about 9°, and increase the midwinter intensity by 3° or 4°; the temperature of spring and autumn being nearly unchanged. But this does not appear to be of itself adequate to the geological effects in question.

It is not our purpose, here, to enter into the inquiry, whether the atmosphere was once more dense than now, whether the earth's axis had once a different inclination to the orbit, or the sun a greater emissive power of heat and light. Neither shall we attempt to speculate upon the primitive heat of the earth nor of planetary space, nor of the supposed connection of terrestrial heat and magnetism; nor inquire how far the existence of coal fields in this latitude, of fossils, and other geological remains have depended upon existing causes. The preceding discussion seems to prove simply that, under the present system of physical astronomy, the sun's intensity could never have been materially different from what is manifested upon the earth at the present day. *The causes of notable geological changes must be other than the relative position of the sun and earth, under their present laws of motion.*

If we extend our view, however, to the general movement of the Sun and Planets in space we find here a *possible* cause for the remarkable changes of temperature traced in the geological periods. For as Poisson conjectured, *Théorie de la Chaleur*, p. 438, the phenomena may depend upon an inequality of temperature in the regions of space, through which the earth has passed. According to a calculation quoted by Prof. Nichol, the velocity of this great movement is six times greater than that of the earth in its orbit, or about 400,000 miles per hour.

In this motion, continued for countless ages, the earth may have traversed the vicinity of some one of the fixed stars, which are suns, whose radiance would tend to efface the vicissitudes of summer and winter, if not of day and night, with a more warm and equable climate. This may have produced those luxuriant forests, of which the present coal fields are the remains; and thus the existence of coal mines in Disco, and other Arctic islands, may be accounted for. If no similar traces exist in the Antarctic zone, the presumption will be strengthened, that the North Pole was presented more directly to the rays of such illuminating sun or star. Indeed, by this position, all possibility of conflict with Neptune, and the other planets which lie nearly in the plane of the ecliptic, was avoided.

The description of such period, with strange constellations and another sun gleaming in the firmament, their mysterious effects upon the growth of animals and vegetation, their untold vicissitudes of light, shadow and eclipse, belong to the romance of astronomy and geology. As in the ancient tradition described by Virgil in the sixth Eclogue:—

> Jamque novum terræ stupeant lucescere solem:
> Altiùs atque cadant submotis nubibus imbres:
> Incipiant silvæ quum primùm surgere, quumque
> Rara per ignotos errent animalia montes.

It is evident that, in receding from the sphere of intensity of such star, as a comet from the sun, the earth's annual temperature would very slowly decrease in process of time, according to the temperature of the space traversed. And, at a remote distance from the stars, the temperature of space ought to remain stationary; as the mean annual temperature of the earth has remained for at least two thousand years past, and without doubt will so continue for ages to come.

SECTION VIII.

ON LOCAL AND CLIMATIC CHANGES OF THE SUN'S INTENSITY.

As the principal topics under this head have been anticipated in the former portions of the work, they need not here be repeated. The inequality of winter, and especially of summer intensities in the northern and southern hemispheres, has already been discussed in the last Section, and ascribed to the changing position of the sun's perigee.

Let us now pass to another local inequality, which consists in the difference of daily intensities at two places situated on the same parallel of latitude, but separated by a considerable interval of longitude. This difference arises solely from hourly change of the Sun's Declination, while moving from the meridian of one place westward to the meridian of the other; the Sun in the interval attaining a higher or lower meridian altitude.

For example, the latitude of Greenwich, near London, is 51° 28′ 39″. Following this parallel west to a point directly north of San Francisco, in California, the difference of longitude is 122° 28′ 2″. At the time of the autumnal equinox, the daily change of the sun's declination is 23′ 23″. Consequently, in passing from the meridian of Greenwich to that of San Francisco, the declination is diminished by $\frac{122° 28' 2''}{360°} \times 23' 23''$, or by 7′ 57″.3.

When the Sun's Declination is 0, at apparent noon at Greenwich, on Sept. 21st, it will be 7′ 57″.3 S. at noon in the longitude of San Francisco on the same day; the semi-diameter being 15′ 59″ or 959″ for Greenwich, and 959″.1 for San Francisco. With these elements, let the sun's daily intensity be computed for both places by formulas (13), (18). The result is 50.13 thermal units for Greenwich, and 49.91 for the place north of San Francisco, on the same latitude. The difference is .22 corresponding to nearly $+ \frac{1}{4}°$ Fahrenheit; and by so much the intensity upon the zenith of Greenwich is greater, on the same day.

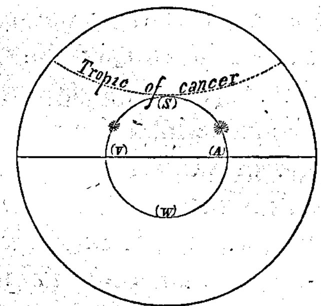
At the vernal equinox, March 20, the sun's daily change of declination would be in the opposite direction, and the difference would become $-\frac{1}{4}°$ F. The inequality of this species thus compensates itself in theory, leaving the *yearly* intensity the same for all places having the same latitude.

For further reference on this point, the daily changes of declination, near the first of each month, are subjoined as follows:—

January,	5′	May,	18′	September,	22′
February,	18′	June,	8′	October,	23′
March,	23′	July,	5′	November,	18′
April,	23′	August,	17′	December,	9′

In this connection, it may be observed that Nervander, Buys Ballot, and Dove have developed a slight inequality of temperature dependent upon the Sun's rotation around his axis, and having the same period of about 27 days; but this result is not confirmed by Lamont, *Poggendorff's Annalen* for 1852.

With respect to maxima and minima, Plate I exhibits a resemblance to two summers and to two winters on the Equator—the sun being vertical at the two equinoxes. On receding from the equator, but still in the torrid zone, the sun will be vertical at equal intervals, before and after the summer solstice, which intervals diminish as the sun approaches the Tropic; the sun being vertical to each locality, when his declination is equal to the latitude of the place; as indicated in the annexed diagram.



On arriving at the Tropic in the yearly motion, the sun can be vertical but once in the year, namely at the summer solstice. At all places more distant from the equator the sun can never be vertical, but will approach nearest this position at the solstice in summer (*s*), and be farthest from it at the solstice of winter (*w*). Thus in the torrid zone, the sun's daily intensity has two maxima and two minima annually; in the temperate zones, one maximum and one minimum; and in the frigid zones, one maximum.

Owing to change of the sun's distance, the intensity is not precisely the same at the autumnal equinox as at the vernal; the difference, however, being small, may here be neglected. And for more full illustration, we exhibit a different projection of the Table in Section IV, showing (Plate IV) the Sun's Diurnal Intensity along the meridian at intervals of thirty days, from June to December, and approximately for the other months. The alternate curves will of course show the sun's changes of intensity in intervals of sixty days. It will be seen that the sun's least yearly range of intensity is not on the Equator, but about 3° of latitude from it north and south. Here the daily heat is most constant, and perpetual summer reigns through the year.

In like manner, the diverging curves show an increasing yearly range, which is greatest in the Polar regions. Also the changes from one day to another are most rapid in spring and autumn. The greatest intensity occurs at the summer solstice, June 21, and the least, at the winter solstice, December 21; so that the yearly range from minimum to maximum is a little wider than the drawn curves indicate. Near the Polar Circle, a singular inflection commences in summer, and the temperature rises rapidly to the Pole.

These laws of Intensity are subject to the retardation in time, mentioned in Section IV, when applied to temperatures; and thus will correspond, generally, with observations. For example, the thermometric column will, during the month of May, rise faster at Quebec than in Florida, and still more rapidly at the Arctic Circle.

It was proved, in Section IV, that the Sun's intensity upon the Pole during eighty-five days in summer, is greater than upon the Equator. Indeed, at the summer solstice it rises to 98.6 thermal units, corresponding nearly to 98° Fahrenheit, which singularly coincides with the temperature of the human body, or blood heat. Though this circumstance may invest the Hyperborean region with new interest, still we cannot assume a brief tropical summer with teeming forms of vegetable and animal life in the centre of the frozen zone. For the measured intensity refers to the outer limit of the atmosphere, upon which the sun shines continually, but from a low altitude which cannot exceed 23° 28′. Much of the heat must, therefore, be absorbed by the air, as happens near the hours of sunrise and sunset in our climate. Also "the vast beds of snow and fields of ice, which cover the land, and the sea in those dreary regions, absorb in the act of thawing or passing to the liquid form, all the surplus heat collected during the continuance of a nightless summer. But the rigor of winter, when darkness resumes her tedious reign, is likewise mitigated by the warmth evolved as congelation spreads over the watery surface." (*Encyc. Brit.*, article Climate.)

The sun's intensity may yet have a somewhat greater effect upon the pole, where it pierces a thinner stratum of the atmosphere than over another portion of the earth's surface. For, in consequence of the centrifugal force of the earth's diurnal motion, the particles of air in all other parts of the earth, being thrown outwards, tend to an increased thickness, in spheroidal strata. We might thence infer that a less proportion of the sun's rays would be absorbed, and a greater portion transmitted through the atmosphere, to the surface of the earth. However this may be in the immediate vicinity of the Pole, yet in the high latitudes hitherto visited by navigators, and which are not nearer than about five or six hundred miles from the North Pole, according to Dr. Kane and others,[1] a dense and lasting fog prevails after the middle of June, through the rest of the summer season, and effectually prevents the rise of temperature which the sun's intensity would otherwise produce.

The question of an *open, unfrozen sea* in the vicinity of the North Pole, has not yet been definitely settled. In this connection we shall only glance at some of the evidences on both sides, without discussing further a subject still unreclaimed from the domain of uncertainty.

"Of this I conceive we may be assured," says Scoresby, Vol. I, p. 46, "that the opinion of an open sea around the Pole is altogether chimerical. We must allow, indeed, that when the atmosphere is free from clouds, the influence of the sun, notwithstanding its obliquity, is, on the surface of the earth or sea, about the time of the summer solstice, greater at the Pole by nearly one-fourth part, than at the equator.[2] Hence it is urged that this extraordinary power of the sun destroys all the ice generated in the winter season, and renders the temperature of the Pole warmer and more congenial to the feelings than it is in some places lying near the equator. Now, it must be allowed, from the same principle, that the influence in the parallel of 78°, where it is computed in the same way to be only about one forty-fifth part

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[1] "The general obscurity of the atmosphere arising from clouds or fogs is such, that the sun is frequently invisible during several successive days. At such times, when the sun is near the northern tropic, there is scarcely any sensible quantity of light from noon to midnight." (*Scoresby's Arctic Regions*, Vol. I, p. 378.)

"The hoar-frost settles profusely in fantastic clusters on every prominence. The whole surface of the sea steams like a lime-kiln, an appearance called the *frost smoke*, caused, as in other instances of the production of vapors, by the waters being still relatively warmer than the incumbent air. At length the dispersion of the mist, and the consequent clearness of the atmosphere announce that the upper stratum of the sea itself has become cooled to the same standard; a sheet of ice quickly spreads, and often gains the thickness of an inch in a single night."

[2] See Section IV (17). The value was first determined by Halley, *Phil. Trans.*, 1693.

less than what it is at the Pole, must also be considerably greater than at the equator. But, from twelve years' observations on the temperature of the icy regions, I have determined the mean annual temperature in latitude 78° to be 16° or 17° F. [that is, about fifteen degrees below freezing point]; how then can the temperature of the Pole be expected to be so very different?"

After some further argument, the author remarks in a note: "Should there be land near the Pole, portions of open water, or perhaps even considerable seas might be produced by the action of the current sweeping away the ice from one side almost as fast as it could be formed. But the existence of land only, I imagine, can encourage an expectation of any of the sea northward of Spitzbergen being annually free from ice."

On the other hand, the following indications in favor of an open sea, are derived from a recent article upon Arctic Researches, announcing that "the existence of the long suspected unfrozen Polar Sea has been all-but proved."

First, it was found that the average annual temperature about the 80th parallel, was higher by several degrees, than that recorded farther south. At the island of Spitzbergen, for example, under the 80th parallel, the deer propagate, and on the northern coast the sea is quite open for a considerable time every year. But at Nova Zembla, five degrees further south, the sea is locked in perpetual ice, and the deer are rarely, if ever seen on its coast. This has led physical geographers to suppose that the milder temperature of Spitzbergen must be attributable to the well-known influence of proximity to a large body of water; while the contiguity of Nova Zembla to the continent was thought to account for the severity of its climate.

Secondly, Captain Parry reached Spitzbergen in May, 1827; from thence he went northward two hundred and ninety-two miles in thirty-five days, during which it rained almost all the time. The ice being much broken, and the current setting toward the south, he could not make way against it, and was compelled to return, which the current greatly facilitated. Besides the current here noticed by Parry, others had been determined before, and more have been ascertained since; so that powerful currents of the Arctic Ocean southward, may be considered as established.

Thirdly, in 1852, Captain Inglefield, while making his summer search for Sir John Franklin, in the northeast of Baffin's Bay, beheld with surprise "two wide openings to the eastward into *a clear and unencumbered sea*, with a distinct and unbroken horizon, which, beautifully defined by the rays of the sun, showed no signs of land, save one island." Further on he remarks, "the changed appearance of the land to the northward of Cape Alexander was very remarkable. South of this cape, nothing but snow-capped hills and cliffs met the eye; but to the northward an agreeable change seemed to have been worked by an invisible agency—here the rocks were of their natural black or reddish-brown color; and the snow which had clad with heavy flakes the more southern shore had only partially dappled them in this higher region, while the western shore was gilt with a belt of ice twelve miles broad, and clad with perpetual snows."

To these may be added the discovery of the southern boundary of an open Polar sea, in the expedition from which Dr. Kane has just returned, October, 1855. "There are facts," observes this distinguished explorer, "to show the necessity and certainty of a vast inland sea at the North. There must be some vast receptacle for the drainage of the Polar regions and the great Siberian Rivers. To prove that water must actually exist, we have only to observe the icebergs. These floating masses cannot be formed without *terra firma*, and it is a remarkable fact that, out of 360°, in only 30° are icebergs to be found, showing that land cannot exist in any considerable portion of the country. Again, Baffin's Bay was long thought to be a close bay, but it is now known to be connected with the Arctic Sea. Within the bay, and covering an area of ninety thousand square miles, there is an open sea from June to October. We find here a vacant space with water at 40° temperature—eight degrees higher than freezing point."[1]

SECTION IX.

ON THE DIURNAL AND ANNUAL DURATION OF SUNLIGHT AND TWILIGHT.

Having thus far considered the intensity of solar radiation upon any part of the earth, we shall lastly pass to examine its duration.

In several publications it has been stated that "the sun is, in the course of the year, the same length of time above the horizon at all places." On applying an accurate analysis, however, it appears, as will presently be shown, that the annual duration of sunlight is subject to a very considerable inequality. This annual inequality increases with the distance from the equator, and is proportional to the sine of the longitude of the sun's perigee.

The longitude of the perigee on Jan. 1, 1850, was 280° 21′ 25″, and increasing at the rate of 61″.47 annually; the sine of the longitude of the perigee is therefore decreasing in value every year, and with it, the inequality of sunlight. At the present time it amounts, in the latitude of 60°, to 36 hours—being additive in the northern, and subtractive in the southern hemisphere. That is, in the latitude of 60° north, the total duration of sunlight in a year is 36 hours more, and in the latitude of 60° south, 36 hours less than on the equator. At either Pole the inequality amounts to 92 hours, or more than seven and a half average days of twelve hours each.

The epoch when the inequality was at its last maximum, is found by dividing the present excess of the longitude of the perigee above three right angles, by the yearly change. The excess, in 1850, was 10° 21′ 25″, which divided by 61″.47 gives a quotient of 606.5 years; which refers back to the period of the middle ages, A. D. 1243.

At a still earlier epoch, this inequality must have entirely vanished. At that

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[1] A reference to Plate IV will confirm what was before known from observations that the extremes of summer and winter temperature range through wider and wider limits from the equator towards each Pole. The application of this general law favors an open Polar sea in summer, as actually seen by explorers, and more recently by Dr. Kane's party in the month of August. But it equally indicates that the sea is frozen over in winter, when there appears no assignable cause, but a calm atmosphere, to mitigate the most intense cold.

epoch, the line of the apsides evidently coincided with the line of the equinoxes, which is computed to have been about 4,000 years before the birth of Christ, at which time chronologists have fixed the first residence of man upon the earth. The luminous year was then of the same length, at all latitudes, from pole to pole.

Though the annual Duration of sunlight thus varies from age to age, and in the northern hemisphere differs from the southern; yet such is the law of the planet's elliptic motion, that the sun's annual Intensity at any latitude north, is precisely the same as at an equal latitude south of the equator. This immediately follows from formula (33), where the annual Intensity is developed in a series of powers of $\cos L$, which is always positive, whether the latitude L be south or north.

Proceeding now to direct investigation, the half day with its augments, may be represented under the general form,

$$H + \text{increase by Refraction} + \text{Twilight}.$$

The first term H is found from the astronomic equation,

$\cos H = -\tan L \tan D = -\dfrac{\tan L \sin \omega \sin T}{\sqrt{1 - \sin^2 \omega \sin^2 T}}$; and this may also take the form[1] of

$$H = \frac{\pi}{2} + \sin^{-1}\left(\frac{\tan L \sin \omega \sin T}{\sqrt{1 - \sin^2 \omega \sin^2 T}}\right). \qquad (43.)$$

Let $u = 2H$, or twice the semi-diurnal arc; then the sum of all the daily values of u through the year, may be found by the method of summation described in Section V. By (22) we have $dx = \dfrac{dT(1-e^2)^{\frac{3}{2}}}{n(1 + e\cos(T-P))^2}$; whence,

$$\int u\,dx = \pi x + \frac{2(1-e^2)^{\frac{3}{2}}}{n}\int_0^{2\pi}\frac{\sin^{-1}(\tan L \tan D)\,dT}{(1 + e\cos(T-P))^2}. \qquad (44.)$$

The general formula of summation, Section V, has the terms $\frac{1}{2}u + \frac{1}{12}\frac{du}{dx} + \ldots$ which in the present case vanish between the limits $T = 0$, and $T = 2\pi$; as will appear from developing u or $2H$ by (43), in terms of $\sin T$. For the annual value therefore, $\Sigma u = \int u\,dx$. Developing the denominator of (44), and substituting for D in the numerator,

$$\Sigma u = \pi x + \frac{2(1-e^2)^{\frac{3}{2}}}{n}\int_0^{2\pi}\sin^{-1}\left(\frac{\tan L \sin\omega \sin T}{\sqrt{1-\sin^2\omega \sin^2 T}}\right)dT\left\{1 - 2e\cos(T-P) + 3e^2\cos^2(T-P) - \ldots\right\}. \qquad (45.)$$

It is evident that $\sin^{-1}$ here would develope in *odd* powers of $\sin T$, which multiplied by dT, and integrated between the limits of 0 and 2π, will vanish, as appears from the formulas of the Integral Calculus; when multiplied by $dT.\cos T$, or $d\sin T$, and integrated between the same limits, they also will vanish, being powers of $\sin T$. Also developing $\cos(T-P)$ into $\cos T\cos P + \sin T \sin P$, and neglecting terms, which would so vanish by integration,

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[1] On this and the following pages, $\sin^{-1}\chi$ denotes the arc whose *sine* is χ; where χ represents any given quantity.

$$\Sigma u = \pi x + \frac{2(1-e^2)^{\frac{3}{2}}}{n}\int_0^{2\pi}\left(H-\frac{\pi}{2}\right)dT\left\{-2e\sin P\sin T - 4e^3(\sin^3 P\sin^3 T + 3\sin P\cos^2 P\sin T\cos^2 T) - \ldots\right\}. \quad (46.)$$

Here $H-\frac{\pi}{2}$ has been substituted for its equal from (43); multiplying the $-\frac{\pi}{2}$ into the following term, and integrating between the limits 0 and 2π, the result vanishes. Also the terms multiplied by $4e^3$ being small; it will be sufficient for them to develope from (43), to the first power, $H-\frac{\pi}{2} = \tan L \sin\omega \sin T - \ldots$ by which their integral is immediately found to be $-4e^3 \tan L \sin\omega \times (\sin^3 P \cdot \frac{3}{4}\pi + 3\sin P\cos^2 P \cdot \frac{\pi}{4})$, or $-3e^3\pi\sin\omega\sin P\tan L$. Besides this, it only remains to integrate the first term depending on $H\sin T\,dT$; but this corresponds to the first term of the formula of annual intensity (23); and if S denote thermal days in the Table of Section V, then $\int_0^{2\pi} H\sin T\,dT = ([\bar{3}.61540]\,S\sec L - E')\,4\cot L\,\mathrm{cosec}\,\omega$; whence finally, converting into hours,

$$\Sigma u = x\,12^h - \frac{16\,e(1-e^2)^{\frac{3}{2}}\sin P}{.2618\,n\sin\omega}\left\{([\bar{3}.61540]\,S\sec L - E')\cot L + \tfrac{3}{8}e^2\pi\sin^2\omega\tan L + \ldots\right\}. \quad (47.)$$

On the equator, L is 0, and the last part vanishes, leaving for the annual duration of sunlight, $x\,12^h$, where x denotes the number of days 365.24.

Therefore $x\,12^h$ represents *the mean value* of the annual duration of sunlight, and the following terms express *the Annual Inequality.* When the latitude is south, both $\cot L$ and $\tan L$ change sign; so that the inequality then becomes negative.

For A. D. 1850, $\sin P$ is negative; substituting the value of this and the other elements for that epoch,

$$\Sigma u = x\,12^h + [2.16700]\times\{[\bar{3}.61540]\,S\sec L - E'\}\cot L + [\bar{3}.88700]\tan L + \ldots \quad (48.)$$

Here brackets include the logarithms of the co-efficients. By this formula the inequality may be readily computed for any latitude between the Equator and the Polar Circle.

In the frigid zone, the summer period of constant day will make another formula necessary. As explained in Section V, the year in that zone may be divided into four periods or intervals. At the beginning of the spring interval, H is 0, and $D = -(90°-L)$; at the end of the spring and beginning of the summer interval, H is 12^h, and $D = 90°-L$; at the end of the summer and beginning of the autumn interval, also, $D = 90°-L$; and at the end of the autumn interval $D = -(90°-L)$. With these data, the equation $\sin D = \sin\omega\sin T$, or $T = \sin^{-1}\left(\frac{\sin D}{\sin\omega}\right)$ enables us to define the lengths of the intervals. Thus the summer interval is measured by the sun's longitude passed over from $T = \sin^{-1}\left(\frac{\cos L}{\sin\omega}\right)$, to $T = \pi - \sin^{-1}\left(\frac{\cos L}{\sin\omega}\right)$, or the length of the arc is $\pi - 2\, sin^{-1}\left(\frac{cos\, L}{sin\, \omega}\right)$. This divided by n, the mean daily motion in longitude, and multiplied by 24, will give the number of hours of sunshine during the summer period, which is $\frac{24}{n}\left(\pi - 2\, sin^{-1}\left(\frac{cos\, L}{sin\, \omega}\right)\right)$.

For the spring and autumn intervals, when day and night alternate, the values must be found by the general formula of summation. Here $u = 2 H = \pi + 2\, sin^{-1}(tan\, L\, tan\, D)$. (49.)

$$d\,u = \frac{2\, tan\, L\, sin\, \omega\, cos\, T\, d\, T}{(1 - sin^2 \omega\, sin^2 T)\sqrt{1 - \frac{sin^2 \omega}{cos^2 L} sin^2 T}}. \qquad (50)$$

$$\frac{1}{d\,x} = \frac{n\,(1 + e\, cos\, P\, cos\, T + e\, sin\, P\, sin\, T)^2}{d\,T(1 - e^2)^{\frac{3}{2}}}.$$

Whence it will be seen that all the terms of $\frac{1}{2}u + \frac{1}{12}\frac{d\,u}{d\,x}$ vanish between the limits of $T = sin^{-1}\left(\frac{cos\, L}{sin\, \omega}\right), sin^{-1}\left(\frac{-cos\, L}{sin\, \omega}\right)$; and $T = sin^{-1}\left(\frac{-cos\, L}{sin\, \omega}\right), sin^{-1}\left(\frac{cos\, L}{sin\, \omega}\right)$; except the following, $\frac{4\, e\, n\, sin\, P\, sin\, \omega\, tan\, L\, cos\, T\, sin\, T}{3\,(1 - sin^2 \omega\, sin^2 T)\sqrt{1 - \left(\frac{sin^2 \omega}{cos^2 L}\right) sin^2 T}}$. Here make $\frac{sin\, \omega}{cos\, L} sin\, T = sin\, Z$, and the expression becomes $\frac{4\, e\, n\, sin\, P\, sin\, L\sqrt{1 - \frac{cos^2 L}{sin^2 \omega} sin^2 Z}\, tan\, Z}{3\,(1 - cos^2 L\, sin^2 Z)}$. And this when taken between the proper limits of $Z = +90°, -90°$, evidently vanishes. It only remains to find $\int u\, d\, x$ between the same limits for the spring and the autumn interval. These limits show that the sun's longitude passed over in the two intervals is $4\, sin^{-1}\left(\frac{cos\, L}{sin\, \omega}\right)$ which divided by n gives the number of days. This multiplied by the first term of u which is π or 12^h, and added to the like result for the summer period, gives $12 \times \frac{2\pi}{n}$ or $x\, 12^h$. And if $sin T = \frac{cos\, L}{sin\, \omega} sin\, Z$, or $sin\, D = cos\, L\, sin\, Z$; then for the whole year, in hours,

$$\Sigma\, u = x\, 12^h + \frac{2\,(1 - e^2)^{\frac{3}{2}}}{.2618\, n}\int_0^{2\pi} sin^{-1}\left(\frac{sin\, L\, sin\, Z}{\sqrt{1 - cos^2 L\, sin^2 Z}}\right) \times \frac{d\, T}{(1 + e\, cos(T - P))^2}. \qquad (51.)$$

It is here assumed that the integral will be taken successively between the limits of $Z = \frac{\pi}{2}, -\frac{\pi}{2}; -\frac{\pi}{2}, +\frac{\pi}{2}$; that is through a whole circumference.

But $d\, T = \frac{cos\, L\, cos\, Z}{sin\, \omega\, cos\, T} d\, Z = \frac{cos\, L}{sin\, \omega} \cdot \frac{cos\, Z\, d\, Z}{\sqrt{1 - \left(\frac{cos\, L}{sin\, \omega}\right)^2 sin^2 Z}}$. As the whole function of Z by which $d\, Z$ is multiplied, would evidently develop in *odd* powers of $sin\, Z$, it follows as in the former operation, that terms which would vanish by integration may be neglected in advance, leaving for the last factor, precisely as in (46), $-2\,e\,\sin P \sin T - 4\,e^3\,(\sin^3 P \sin^3 T + 3 \sin P \cos^2 P \sin T \cos^2 T) - \ldots$

Substituting here for $\sin T$ its equal $\frac{\cos L}{\sin \omega} \sin Z$; the first term of the product is

$$-2\,e \sin P\,.\,\sin^{-1}\left(\frac{\sin L \sin Z}{\sqrt{1-\cos^2 L \sin^2 Z}}\right).\frac{\cos^2 L}{\sin^2 \omega}\,\frac{\sin Z \cos Z\,d\,Z}{\sqrt{1-\frac{\cos^2 L}{\sin^2 \omega}\sin^2 Z}}.$$

Integrating this by parts, we have

$$2\,e \sin P.\sin^{-1}\left(\frac{\sin L \sin Z}{\sqrt{1-\cos^2 L \sin^2 Z}}\right)\sqrt{1-\frac{\cos^2 L}{\sin^2 \omega}\sin^2 Z}$$

$$-2\,e \sin P \int \frac{\sin L \sqrt{1-\frac{\cos^2 L}{\sin^2 \omega}\sin^2 Z}.\,d\,Z.}{1-\cos^2 L \sin^2 Z}$$

Multiplying both numerator and denominator of the last term by the radical, it takes the form of $\int \frac{\left(\sin L - \frac{\sin L \cos^2 L \sin^2 Z}{\sin^2 \omega}\right)}{(1-\cos^2 L \sin^2 Z)\sqrt{1-\frac{\cos^2 L}{\sin^2 \omega}\sin^2 Z}}$, or $\int \frac{\sin L\,d\,Z}{\sin^2 \omega \sqrt{1-\frac{\cos^2 L}{\sin^2 \omega}\sin^2 Z}}$ $+\left(\sin L - \frac{\sin L}{\sin^2 \omega}\right)\Pi$; that is, $\frac{\sin L}{\sin^2 \omega}.\,F - \sin L \cot^2 \omega.\,\Pi$; where the letters F and Π designate elliptic functions. When integrated between the above named limits, or an entire circumference, the former term of the integral vanishes, leaving only $-\frac{8\,e \sin P \sin L}{\sin^2 \omega}\,(F' - \cos^2 \omega.\,\Pi')$.

Developing only to the first power, the next term to be integrated is $-\frac{4\,e^3 \sin^3 P \cos^4 L}{\sin^4 \omega} \sin L \sin^4 Z\,d\,Z$, which between 0 and $2\,\pi$, gives $-3\,e^3\,\pi \sin^3 P \sin L\,.\,\frac{\cos^4 L}{\sin^4 \omega}$. As the remaining terms are still smaller they may be omitted; whence

$$\Sigma\,u = x\,12^h - \frac{16\,e\,(1-e^2)^{\frac{3}{2}} \sin P}{.2618\,n \sin \omega}\left\{\frac{\sin L}{\sin \omega}F' - \frac{\sin L \cos^2 \omega}{\sin \omega}\Pi' + \frac{3}{8}\frac{e^2\,\pi \sin^2 P \sin L \cos^4 L}{\sin^3 \omega} - \ldots\right\} \qquad (52.)$$

The multiplier for converting $\Sigma\,u$ in Section V into thermal days S, is $\frac{n\sqrt{1-e^2}}{4\,c} \times \frac{365.24}{1.5065}$; whence by (28), $-\frac{\sin^2 L \cos^2 \omega\,\Pi'}{\sin \omega} = \frac{1.5065\,S}{365.24} - \frac{\cos^2 \omega}{\sin \omega}F' - \sin \omega.\,E'$; and substituting this, we have for the year 1850,

$$\Sigma u = x 12^h + [2.16700]\frac{\sin \omega}{\sin L}\left\{[\bar{3}.61540]\frac{S}{\sin \omega} - E' + \left(1-\frac{\cos^2 L}{\sin^2 \omega}\right)F'\right\}$$
$$+ [\bar{1}.87213] \sin L \cos^4 L + \ldots \qquad (53.)$$

Here the modulus or eccentricity of the elliptic quadrant is $\frac{\cos L}{\sin \omega}$; and the brackets denote logarithms of the co-efficients. Such is the formula for the Frigid Zone.

By means of equations (48), (53), I have computed the annual duration of sunlight $\Sigma (2H)$, according to the rising and setting of the sun's centre, without regard to refraction. It is the half of 365.24 days, or 182.62 days increased by the quantities in the following table, for the northern hemisphere, and diminished by the same for the southern hemisphere:—

Annual Inequality of Sunlight.

Latitude.	Inequality.	Latitude.	Inequality.
0°	0 h. 00 m.	50°	24 h. 08 m.
10	3 25	60	36 51
20	7 07	70	66 52
30	11 23	80	86 02
40	16 40	90	92 01

Having thus discussed the duration of Sunlight, let us next consider its increase by Refraction, and by Twilight. The mean horizontal refraction, according to Mr. Lubbock's result, is 2075″, or 34′35″; the barometer standing at 30 inches, and the thermometer at 50° F. But as this is somewhat greater than what has been usually employed, we shall adopt 34′ as the mean value for determining the increase of daylight by direct refraction.

With respect to the duration of Twilight, A. Bravais, who has made extensive observations upon the phenomenon, observes, in the *Annuaire Météorologique de la France* for 1850, p. 34: "The length of twilight is an element useful to be known: by prolonging the day, it permits the continuance of labor. Unfortunately, philosophers are not agreed upon its duration. It depends on the angular quantity by which the sun is depressed below the horizon; but it is also modified by several other circumstances, of which the principal is the degree of serenity of the air. Immediately after the setting of the sun, the curve which forms the separation between the atmospheric zone directly illuminated by the sun, and that which is only illuminated secondarily, or by reflection, receives the name of the *crepuscular curve*, or *Twilight Bow*.[1] Some time after sunset, this bow, in traversing the heavens from east to west, passes the zenith; this epoch forms the end of *Civil Twilight*, and is the moment when planets and stars of the first magnitude begin to be visible. The eastern half of the heavens being then removed beyond solar illumination, night commences to all persons in apartments whose windows open to the east. Still later the Twilight bow itself disappears in the western horizon; it is then the end of *Astronomic Twilight;* it is closed night. We may estimate that civil twilight ends, when the sun has declined 6° below the horizon; and that a decline of 16° is necessary to terminate the astronomic twilight.

"I depart here from the general opinion, which fixes at 18° the solar depression at the end of twilight, and at 9° that which characterizes the end of civil twilight.

[1] The phenomenon is equally conspicuous in the west, before the rising of the sun, and in certain states of the atmosphere is scarcely less beautiful than the rainbow, for the symmetry and vivid tinting of its colors.

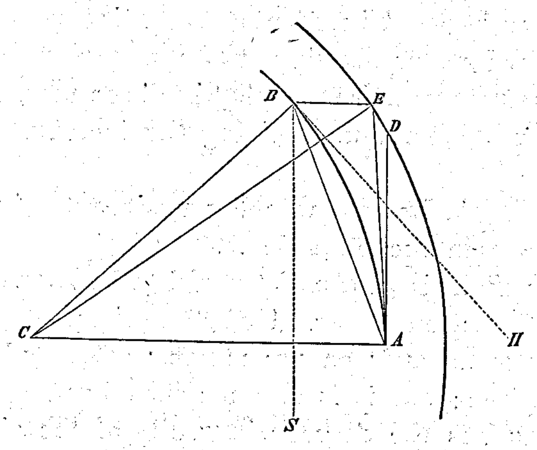
The numbers which I have adopted are derived from numerous observations." "The shortest civil twilight takes place on the 29th of September, and on the 15th of March; the longest on the 21st of June. The shortest astronomic twilight occurs on the 7th of October, and on the 6th of March; the longest on the 21st of June, in this latitude. Above the 50th degree of latitude, twilight lasts through the whole night at the summer solstice."

The analytic solution of the problem to find the time of *shortest twilight* was first given by John Bernoulli; the result, found in various works, is expressed by the two equations,

$$\sin\frac{t}{2} = \frac{\sin\frac{1}{2}m}{\cos lat}; \quad \sin Dec. = -\tan\tfrac{1}{2}m \sin lat.$$

Here t denotes the duration of shortest twilight, and m is the sun's depression 16° or 18° below the horizon.

To pursue the discussion of the physical details of twilight, would occupy too much space, and we shall here only glance at the method of Lambert for determining the height of the atmosphere from twilight. The demonstration is based upon the examples solved in *Gehler's Physikalisches Wörterbuch.*



Lambert found that when the true depression of the sun below the horizon was 8° 3′, (b), the height of the twilight arch was 8° 30, (a); and when the true depression of the sun was 10° 42′, the altitude of the bow was 6° 20′.

In the figure let C denote the centre and $A B$ the surface of the earth; $D E$, the outer limit of the atmosphere; B, the place of the observer; and E, the position of the bow.

Let r denote the refraction due to the altitude (a), then the angle $E B H = a - r = a'$. When the sun's centre is apparently in the horizon of A, it is really about 34′ below it. Denoting this horizontal refraction by r', and deducting it from b, leaves the angle $H B S = b - r' = b'$; where $B S$ is parallel to the tangent $A D$. But the ray of light after passing A is further refracted by r' or 34′ to E.

The angle $H B S$ or b', according to a proposition of geometry, is measured by the arc $A B$; whence the angle $A C B = b'$; then drawing the chord $A B$, and denoting the earth's radius by R, the isosceles triangle $A C B$ gives $A B = 2 R \sin\frac{1}{2} b'$.

In triangle $A B E$, the angle A is evidently $\frac{1}{2} b - r'$; and deducting the other three angles of the quadrilateral from four right angles, leaves the angle $E = 180° - a' - b' + r'$; then,

$$\sin E \text{ or } \sin(a' + b' - r') : \sin(\tfrac{1}{2}b' - r') :: AB : BE = \frac{2R\sin\frac{1}{2}b' . \sin(\frac{1}{2}b' - r')}{\sin(a' + b' - r')}. \quad (54.)$$

In the triangle $C B E$, two sides and the included angle $90° + a'$ are now given to find the third side CE; from which deducting R, leaves the required height of the atmosphere.

With this mode of calculation, the first observations of Lambert, before stated, determine the height to be 17 miles; and the second observations, 25 miles. And a still later observation would have given a still greater height, owing, perhaps, to the mingling of direct and reflected rays. The subject awaits further improvement; though some extensions have been made by M. Bravais, in the *Annuaire Météorologique de la France* for 1850.

If we regard only the appearance of the Twilight bow, the limits of the sun's depression assigned by M. Bravais are doubtless nearly correct, namely 16° for astronomical, and 6° for civil twilight. But, regarding only the actual intensity of light falling upon the eye, it appears that the effects of the bow are further increased by indefinite reflection among the particles of air, and this may increase the average limits to 9° for civil, and 18° for astronomical twilight. Without determining which view ought to be adopted, a mean has here been taken, and *the following tables have been calculated on the assumption that the sun is* 7½° *below the horizon at the end of civil twilight; and* 17°, *at the end of astronomic twilight.*

The increase of the day by Refraction and by the twilights, may all be comprehended in one general formula. Let m denote the sun's depression below the horizon at the end of either period; then the distance from the Pole to the zenith, $90° - L$, the distance from the Pole to the sun, $90° - D$, the distance from the zenith to the sun $90° + m$, or three sides of a spherical triangle are given to find the hour angle $H + T$, as in the following equation:—

$$\cos(H+\tau) = \frac{-\sin L \sin D - \sin m}{\cos L \cos D} = + \cos H - \frac{\sin m}{\cos L \cos D}. \qquad (55.)$$

Here τ denotes the increase by refraction or by Twilight, according as m is taken at 34′, at 7½°, or 17°.

When twilight lasts through the whole night, it is evident that at the commencement and at the end of such period, $\tau = 12^h - H$. Substituting this value in (55),

$$-1 = \frac{-\sin L \sin D - \sin m,}{\cos L \cos D}, \text{ or } \cos(L+D) = \sin m; \text{ that is, } D = 90° - L - m. \quad (56.)$$

The corresponding yearly limit for constant sunlight has already been found to be indicated by $D = 90° - L$. The lowest latitude where this is possible is evidently $L = 90° - \omega$, or at the Polar Circle. In like manner, the lowest latitude where twilight through the whole night occurs, is $L = 90° - \omega - m = 49° 32'$ north or south of the equator.

During the long night in the Polar regions, twilight will be, for a time, impossible; that is, so long as the sun continues more than 17° below the horizon. The limits of this period will be defined by making $H + \tau$ equal to 0, in (55); whence $L - D = 90° + m$, or $D = -90° - m + L$. (57.)

The corresponding yearly limit of sunlight is indicated by $D = -90° + L$. But the application of these limits is reserved till after an expression for the annual duration of twilight has been found by the method of summation described in Section V. For this purpose, equation (55) may be put under the form of

$$\tau = -H + \cos^{-1}\left(\cos H - \frac{\sin m}{\cos L \cos D}\right). \qquad (58.)$$

Developing in powers of *sin m* by Maclaurin's Theorem,

$$\left.\begin{aligned}\tau = \frac{\sin m}{\sqrt{\cos^2 L - \sin^2 D}} - \frac{\frac{1}{2}\sin^2 m \sin L \sin D}{(\cos^2 L - \sin^2 D)^{\frac{3}{2}}} + \tfrac{1}{6}\sin^3 m \frac{(\cos^2 L + (3\sin^2 L - 1)\sin^2 D)}{(\cos^2 L - \sin^2 D)^{\frac{5}{2}}} \\ - \tfrac{1}{8}\sin^4 m \left(\frac{3 \sin L \sin D}{(\cos^2 L - \sin^2 D)^{\frac{5}{2}}} + \frac{5 \sin^3 L \sin^3 D}{\cos^2 L - \sin^2 D)^{\frac{7}{2}}}\right) + \ldots\end{aligned}\right\} \quad (59.)$$

With respect to the yearly limits already assigned, (55), (56), (57), we know that in the lower latitudes, twilight recurs regularly, while the sun's longitude T varies from zero to an entire circumference; but in the Polar zone, this continuity is interrupted. Still, in integrating for the yearly duration of twilight between the proper limits, $\frac{1}{2}u + \frac{1}{12}\frac{du}{dx}$ being expressed in terms of $\sin D$ or $\sin T$ will vanish, even in the Polar zone, leaving only $\int u\,dx$. And with respect to dx, since $\cos T\,dT$ is $d \sin T$, which multiplied into the development of τ, would integrate in powers of $\sin T$ which vanish, we may reject all such factors in advance, leaving,

$$dx = \frac{(1-e^2)^{\frac{3}{2}}}{n} dT[1 - 2e \sin P \sin T + 3e^2(\cos^2 P - \cos 2P \sin^2 T) + \ldots]. \quad (60.)$$

Were this multiplied into (59), making $u = 2\tau$, and substituting for $\sin D$ its equal $\sin \omega \sin T$, then integrating between the proper limits, and dividing by $\frac{\pi}{12}$ in order to convert arc into hours of time, we should obtain the annual duration of twilight expressed in elliptic functions. It will be more convenient, however, to resort to circular functions.

To obtain the duration of Twilight in another form, let N denote the interval of *Night*, from the end of the evening twilight to midnight, or from midnight to the morning twilight, computed by the sun's midnight declination. The duration of N will correspond to any assumed depression or elevation of the crepusculum circle, or to any compatible value of m. Then $N = 12^h - (H + \tau)$;

$$\cos N = -\cos(H + \tau) = \frac{\sin L \sin D + \sin m}{\cos L \cos D}.$$

$$\frac{dN}{d \sin D} = \frac{-\sin L - \sin m \sin D}{\cos L \cos^3 D \sin N} = \frac{-\sin L - \sin m \sin D}{\cos^2 D \sqrt{\cos^2 L - \sin^2 m - \sin^2 D - 2 \sin L \sin m \sin D}}.$$

Developing $\cos^2 D$ into the numerator under the form of $(1 - \sin^2 D)^{-1}$; also resolving the radical into two factors, one of which is $\sqrt{\cos^2 L - \sin^2 m}$, and developing the other into the numerator to the fifth power of $\sin D$; then multiplying the factors, and employing Maclaurin's Theorem, or integrating; also making $\cos^2 L - \sin^2 m = s$;

$$N = \cos^{-1}\left(\frac{\sin m}{\cos L}\right) - \frac{\sin L \sin D}{\sqrt{s}} - \frac{\frac{1}{2}\sin m \cos^2 m \sin^2 D}{s^{\frac{3}{2}}} - \frac{\sin L}{6 s^{\frac{3}{2}}}\left(\cos 2L + 2 + \frac{3 \sin^2 L \sin^2 m}{s}\right)\sin^3 D - \frac{\sin m}{8 s^{\frac{3}{2}}}\left(\cos 2m + 2 + \frac{3 \sin^2 L(1 + \sin^2 m)}{s} + \frac{5 \sin^4 L \sin^2 m}{s^2}\right)\sin^4 D$$

$$- \frac{\sin L}{10 s^{\frac{3}{2}}}\left\{\cos 2L + 2 + \frac{3 \sin^2 m (1 + \sin^2 L) + \frac{3}{4}}{s} + \frac{5 \sin^2 L \sin^2 m (\sin^2 m + \frac{3}{2})}{s^2} + \frac{\frac{35}{4} \sin^4 L \sin^4 m}{s^3}\right\}\sin^5 D - \frac{\sin m}{12 s}\left\{\cos 2m + 2 + \frac{3 \sin^2 L (1 + \sin^2 m) + \frac{3}{4}}{s} + \right.$$

$$\frac{5\, sin^2 L\, (sin^2 m\, (sin^2 L + \frac{3}{2}) + \frac{3}{4})}{s^2} + \frac{35\, sin^4 L\, sin^2 m\, (\frac{1}{2} + \frac{1}{4}\, sin^2 m)}{s^3} + \frac{315\, sin^6 L\, sin^4 m}{20\, s^4} \Big\} sin^6 D - \ldots \quad (62.)$$

Here let us put $cos\, N' = \frac{sin\, m}{cos\, L}$; and denoting the co-efficients of $sin\, D, sin^2 D \ldots$ by $-N_1, -N_2, \ldots$ we have,

$$N = N' - N_1\, sin\, D - N_2\, sin^2 D - N_3\, sin^3 D - N_4\, sin^4 D - \ldots$$

Multiplying this by the former series for $d\,x$, and integrating the products, after substituting $sin\,\omega\, sin\, T$ for $sin\, D$, and dividing by $\frac{\pi}{12}$,

$$\begin{aligned}
\Sigma\, 2\, N = \frac{24^h (1-e^2)^{\frac{3}{2}}}{n\,\pi} \Big\{ & N'\,[T(1 + 3e^2 cos^2 P) + 2\, e\, sin\, P\, cos\, T - 3\, e^2 cos\, 2\, P \textstyle\int_2 + \ldots] \\
& + sin\,\omega\, N_1\, [(1 + 3\, e^2 cos^2 P)\, cos\, T + 2\, e\, sin\, P \textstyle\int_2 + 3\, e^2 cos\, 2\, P \int_3 + \ldots] \\
& - \frac{sin^2\omega}{1.2} N_2\, [(1 + 3\, e^2 cos^2 P) \textstyle\int_2 - 2\, e\, sin\, P \int_3 - 3\, e^2 cos\, 2\, P \int_4 - \ldots] \\
& - \frac{sin^3\omega\, N_3}{1.2.3}\, [(1 + 3\, e^2 cos^2 P) \textstyle\int_3 - 2\, e\, sin\, P \int_4 - 3\, e^2 cos\, 2\, P \int_5 - \ldots] \quad (63.) \\
& - \frac{sin^4\omega\, N_4}{1.2.3.4}\, [(1 + 3\, e^2 cos^2 P) \textstyle\int_4 - 2\, e\, sin\, P \int_5 - 3\, e^2 cos\, 2\, P \int_6 - \ldots] \\
& - \frac{sin^5\omega\, N_5}{1.2.3.4.5}\, [(1 + 3\, e^2 cos^2 P) \textstyle\int_5 - 2\, e\, sin\, P \int_6 - 3\, e^2 cos\, 2\, P \int_7 - \ldots] \\
& - \frac{sin^6\omega\, N_6}{1.2.3.4.5.6}\, [(1 + 3\, e^2 cos^2 P) \textstyle\int_6 - 2\, e\, sin\, P \int_7 - 3\, e^2 cos\, 2\, P \int_8 - \ldots] \ldots + C \Big\}.
\end{aligned}$$

The integral signs here designate the following quantities:—

$$\begin{aligned}
\textstyle\int_2 &= \int sin^2\, T\, d\, T = -\frac{sin\, 2\, T}{4} + \frac{T}{2}. \\
\textstyle\int_3 &= \int sin^3\, T\, d\, T = \frac{cos\, 3\, T}{12} - \frac{3\, cos\, T}{4}. \\
\textstyle\int_4 &= \int sin^4\, T\, d\, T = \frac{sin\, 4\, T}{32} - \frac{sin\, 2\, T}{4} + \frac{3\, T}{8}. \quad (64.) \\
\textstyle\int_5 &= \int sin^5\, T\, d\, T = -\frac{cos\, 5\, T}{80} + \frac{5\, cos\, 3\, T}{48} - \frac{5\, cos\, T}{8}. \\
\textstyle\int_6 &= -\frac{sin\, 6\, T}{192} + \frac{3\, sin\, 4\, T}{64} - \frac{15\, sin\, 2\, T}{64} + \frac{5\, T}{16}. \ldots
\end{aligned}$$

In the year 1850, $\omega = 23° 27\frac{1}{2}'$; $P = 280° 22\frac{1}{2}'$; $e = .01676$; $n = \frac{2\pi}{365.24}$; $1 + 3\, e^2 cos^2 P$ 1.000027; $2\, e\, sin\, P = -.032972$; $3\, e^2 cos\, 2\, P = -.000788$; $\frac{24^h (1-e^2)^{\frac{3}{2}}}{n.\pi} = 443^h.89$.

For the lower and middle latitudes, where $2\, N$ and $2\, (H + \tau)$ alternate in every twenty-four hours through the year, we may integrate through an entire circumference. In this case, equation (63) is materially simplified; and denoting by brackets the common logarithms of the co-efficients,

$$\left.\begin{aligned}
\Sigma\, 2 N = [3.44564]\, N' - [1.26253]\, N_1 - [2.04360]\, N_2 - [\bar{1}.55940]\, N_3 \\
- [0.03944]\, N_4 - [\bar{3}.37930]\, N_5 - [\bar{3}.68312]\, N_6 - \ldots
\end{aligned}\right\}. \quad (65.)$$

At the Pole, the duration of twilight is easily found by noting in the ephemeris the time at which the sun's declination south, is equal to the depression of the crepusculum circle below the horizon; this instant and the equinox being its limits of duration. As before indicated, the limit of refractional light is when the sun is 34′ below the horizon, or $m = 34'$; civil twilight, when $m = 7\frac{1}{2}°$; and common or astronomical twilight when $m = 17°$. Thus we shall find,

Annual Duration.

	Sunlight. $\Sigma\,(2\,H)$.	Refractional Light.	Civil Twilight.	Astronomic Twilight.	Darkness. $\Sigma\,(2\,N)$.
North Pole.	186d. 11h.	2d. 22h.	38d. 15h.	94d. 16h.	84d. 3h.
Lat. 40°.	183d. 8h.	1d. 14h.	21d. 6h.	49d. 2h.	132d. 20h.
Equator.	182d. 15h.	1d. 5h.	15d. 21h.	36d. 1h.	146d. 14h.

From this table, it appears that the annual length of darkness diminishes from the equator to the pole; while the duration of twilight increases from about one month on the equator to three months at the Pole. In this latitude, about thirty-eight hours of daylight, at the sun's rising and setting, are annually due to atmospheric refraction. The second, fifth, and sixth columns correspond to the formula $\Sigma\,(2\,H) + \Sigma\,(2\,\tau) + \Sigma\,(2\,N) = 365^{d}\,6^{h}$.

In further illustration of this subject, the duration from noon to midnight, or from midnight to noon, of Sunlight, Astronomic Twilight, and Darkness are exhibited to the eye in the accompanying Plate V, for every day in the year, on different latitudes. On the equator, it will be seen that Twilight has its least value, and is almost uniform through the year. In the latitude of 40°, the limiting curves of twilight bend upward in an arch-like form. The upper curve at the same time recedes from the lower, and encroaches upon the duration of darkness, till, as shown for latitude 60°, twilight lasts through the whole night in summer. If the first and last extremities of the curves at January and December be united to complete the circuit of a year, darkness there, will be represented by an elliptic segment; the longest nights and shortest days being at mid-winter. In approaching the highest latitudes, the lines which form the limits continually change their inclination, till at the Pole, they become perpendicular to their position at the Equator.

The present Section contains formulæ and tables for determining both the diurnal and the yearly limits of twilight, with tabular examples for A. D. 1853, computed for 34′, 7° 30′, and 17°, depressions of the crepusculum circle below the horizon; the reasons for which have before been stated. Although these phenomena are varied by mists and clouds, and by the atmospheric temperature and density, still the assumption of mean depressions, has been necessary in order to obtain a general view of their laws of continuance. The duration of moonlight which is unattended by sensible heat, has not been discussed. From this source, the reign of night is still further diminished, till in this latitude, the remaining duration of total darkness after twilight and moonlight, can scarcely exceed three months in the year. The interval towards the close of astronomic or common twilight, corresponds to what is commonly termed, in the country, "early candle-light," when the glimmering landscape fades on the sight, and the stars begin to be visible. The end of civil twilight marks the time at which some city corporations in Europe are said to have made regulations for lighting the street lamps.

In conclusion, without entering into further details, the connection of solar heat and light has enabled us to exhibit, by the same formulæ and curves, the intensities of both in common. Indeed so close is the analogy that even the monthly height of the mercurial column, which shows the temperature, indicates generally the average intensity of sunlight in that locality.

Half Days, or Semi-Diurnal Arcs, in the Northern Hemisphere.

1853.	Lat. 0°.	Lat. 10°.	Lat. 20°.	Lat. 30°.	Lat. 40°.	Lat. 50°.	Lat. 60°.	Lat. 70°.	Lat. 80°.	Lat. 90°.
	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.
Jan. 1	6 00	5 43	5 24	5 03	4 37	3 58	2 51	0 00	0 00	0 00
" 16	6 00	5 44	5 28	5 09	4 45	4 11	3 13	0 00	0 00	0 00
" 31	6 00	5 48	5 35	5 18	5 00	4 33	3 49	2 04	0 00	0 00
Feb. 15	6 00	5 51	5 41	5 30	5 17	4 58	4 29	3 29	0 00	0 00
Mar. 2	6 00	5 55	5 50	5 44	5 36	5 26	5 10	4 40	3 00	0 00
" 17	6 00	5 59	5 58	5 57	5 56	5 54	5 51	5 46	5 32	0 00
April 1	6 00	6 04	6 07	6 11	6 16	6 22	6 32	6 51	7 49	12 00
" 16	6 00	6 07	6 15	6 24	6 35	6 50	7 12	7 59	12 00	12 00
May 1	6 00	6 11	6 22	6 36	6 53	7 15	7 52	9 12	12 00	12 00
" 16	6 00	6 14	6 29	6 46	7 08	7 38	8 28	10 50	12 00	12 00
" 31	6 00	6 16	6 34	6 54	7 19	7 55	8 58	12 00	12 00	12 00
June 15	6 00	6 18	6 36	6 58	7 25	8 04	9 14	12 00	12 00	12 00
July 1	6 00	6 17	6 36	6 57	7 23	8 02	9 09	12 00	12 00	12 00
" 16	6 00	6 16	6 33	6 52	7 17	7 51	8 51	12 00	12 00	12 00
" 31	6 00	6 13	6 28	6 44	7 04	7 32	8 19	10 20	12 00	12 00
Aug. 15	6 00	6 10	6 21	6 33	6 48	7 09	7 42	8 53	12 00	12 00
" 30	6 00	6 06	6 13	6 21	6 31	6 44	7 04	7 43	10 14	12 00
Sept. 14	6 00	6 02	6 05	6 08	6 11	6 16	6 23	6 37	7 18	12 00
" 29	6 00	5 59	5 57	5 54	5 52	5 48	5 43	5 33	5 03	0 00
Oct. 14	6 00	5 54	5 48	5 41	5 32	5 20	5 02	4 27	2 20	0 00
" 29	6 00	5 51	5 40	5 28	5 13	4 53	4 22	3 14	0 00	0 00
Nov. 13	6 00	5 47	5 33	5 17	4 57	4 29	3 43	1 46	0 00	0 00
" 28	6 00	5 44	5 27	5 08	4 43	4 09	3 09	0 00	0 00	0 00
Dec. 13	6 00	5 43	5 24	5 03	4 37	3 57	2 49	0 00	0 00	0 00

Increase of the Half Day at Sunrise, or Sunset, by Refraction.

1853.	Lat. 0°.	Lat. 10°.	Lat. 20°.	Lat. 30°.	Lat. 40°.	Lat. 50°.	Lat. 60°.	Lat. 70°.	Lat. 80°.	Lat. 90°.
	m.	m.	m.	m.	m.	m.	m.	m.	m.	m.
January	2.5	2.5	2.6	2.8	3.3	4.4	6.7	0.00	0.00	0.00
February	2.4	2.4	2.5	2.7	3.1	3.8	5.1	9.0	0.00	0.00
March	2.3	2.3	2.5	2.7	3.0	3.8	4.6	7.0	14.0	0.00
April	2.3	2.4	2.5	2.8	3.2	3.8	5.0	8.0	0.00	0.00
May	2.4	2.5	2.6	3.2	3.5	4.5	6.1	22.0	0.00	0.00
June	2.5	2.6	2.8	3.1	3.7	4.9	7.6	0.00	0.00	0.00
July	2.5	2.5	2.7	3.0	3.5	4.7	6.7	0.00	0.00	0.00
August	2.4	2.5	2.5	2.8	3.2	4.0	5.2	9.7	0.00	0.00
September	2.3	2.4	2.5	2.7	3.1	3.7	4.6	7.0	14.7	0.00
October	2.3	2.4	2.5	2.7	3.1	3.7	4.9	7.5	24.3	0.00
November	2.4	2.5	2.6	2.8	3.2	3.9	5.9	16.3	0.00	0.00
December	2.5	2.5	2.7	2.9	3.5	4.6	7.5	0.00	0.00	0.00

Duration of Civil Twilight, Morning or Evening.

1853.	Lat. 0°.	Lat. 10°.	Lat. 20°.	Lat. 30°.	Lat. 40°.	Lat. 50°.	Lat. 60°.	Lat. 70°.	Lat. 80°.	Lat. 90°.
	m.	m.	m.	m.	m.	m.	h. m.	h. m.	h. m.	h. m.
January	32	33	34	37	43	57	1 16	3 21[2]	0 00	0 00
February	31	31	32	35	40	49	1 15	1 40	4 01[2]	0 00
March	30	30	32	35	39	50	1 03	1 29	3 04	12 00[2]
April	30	31	33	36	41	50	1 08	2 09	0 00	0 00
May	32	33	34	42	45	58	1 37	1 10[1]	0 00	0 00
June	33	34	36	40	48	64	2 46[1]	0 00	0 00	0 00
July	32	33	35	39	46	61	2 03	0 00	0 00	0 00
August	31	32	33	36	42	52	1 15	3 07[1]	0 00	0 00
September	30	31	32	35	40	47	1 02	1 35	4 42[1]	0 00
October	30	31	32	35	40	47	1 01	1 31	3 26	12 00[2]
November	31	32	34	37	42	51	1 10	2 16	0 00	0 00
December	33	33	35	38	44	60	1 22	2 42[2]	0 00	0 00

Duration of Astronomical Twilight, Morning or Evening.

1853.	Lat. 0°.	Lat. 10°.	Lat. 20°.	Lat. 30°.	Lat. 40°.	Lat. 50°.	Lat. 60°.	Lat. 70°.	Lat. 80°.	Lat. 90°.
	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.
January	1 13	1 13	1 17	1 24	1 39	1 56	2 38	5 29[2]	4 35[2]	0 00
February	1 10	1 10	1 14	1 20	1 30	1 43	2 20	3 32	7 49[2]	12 00[2]
March	1 08	1 09	1 12	1 19	1 30	1 48	2 21	3 44	6 29[1]	12 00[2]
April	1 09	1 11	1 15	1 24	1 36	2 01	3 06	4 01[1]	0 00	0 00
May	1 12	1 14	1 19	1 29	1 48	2 37	3 33[1]	1 10[1]	0 00	0 00
June	1 14	1 17	1 23	1 35	1 59	3 56[1]	2 46[1]	0 00	0 00	0 00
July	1 13	1 16	1 21	1 32	1 54	2 59	3 09[1]	0 00	0 00	0 00
August	1 10	1 12	1 16	1 25	1 40	2 11	4 18[1]	3 07[1]	0 00	0 00
September	1 08	1 09	1 13	1 18	1 31	1 51	2 30	5 23[1]	4 42[1]	0 00
October	1 09	1 10	1 13	1 19	1 29	1 47	2 18	3 25	7 48	12 00[2]
November	1 12	1 12	1 15	1 22	1 33	1 52	2 29	4 14	5 43[2]	0 00
December	1 14	1 15	1 18	1 25	1 37	2 00	2 47	5 03[2]	3 33[2]	0 00

[1] Twilight through the whole night. [2] Twilight without day.

NOTE.—Astronomical Twilight includes the duration of Civil Twilight.

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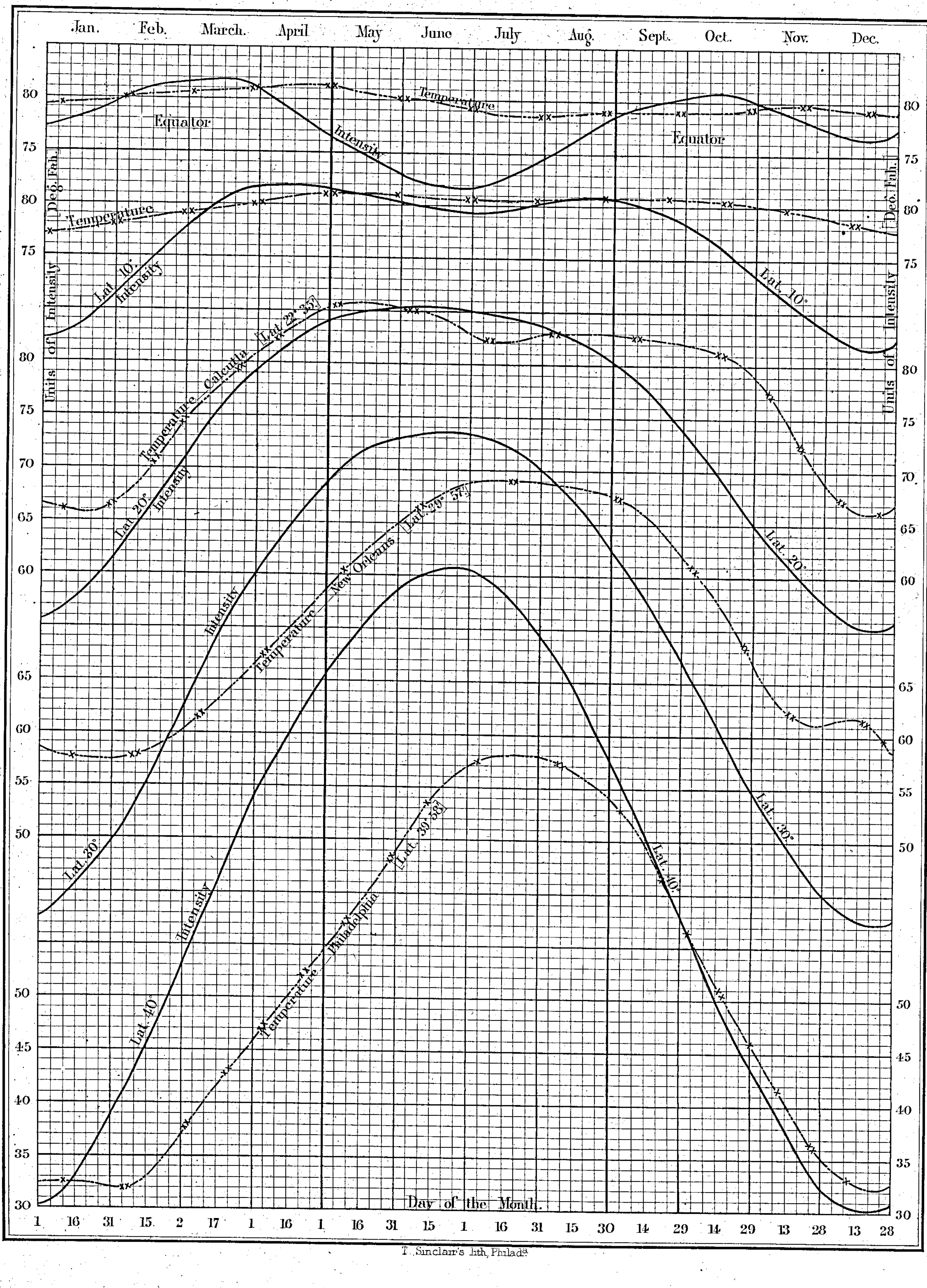
NOVEMBER, 1856.

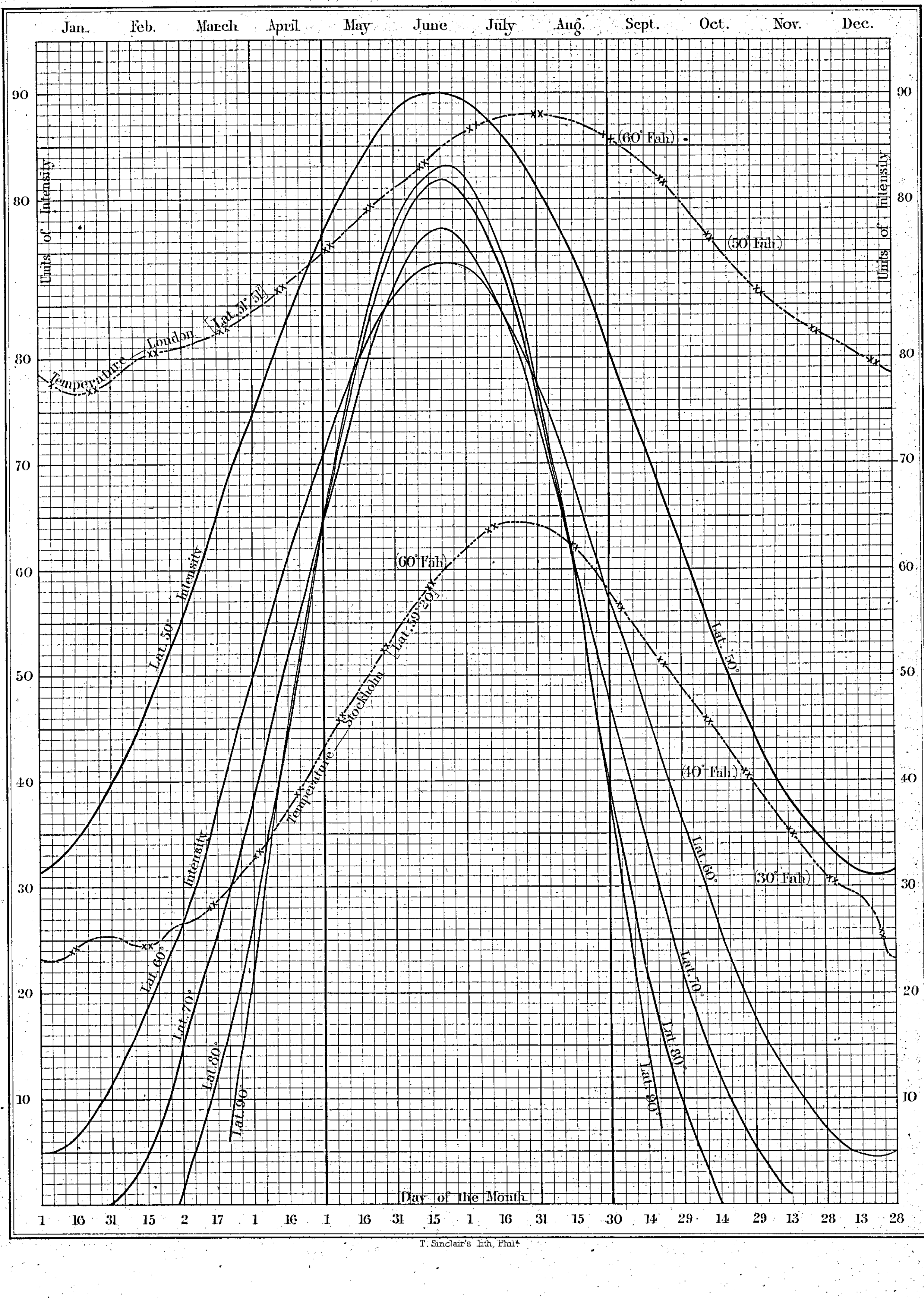
THE SUN'S DIURNAL INTENSITY.

I.



T. Sinclair's lith. Philada.

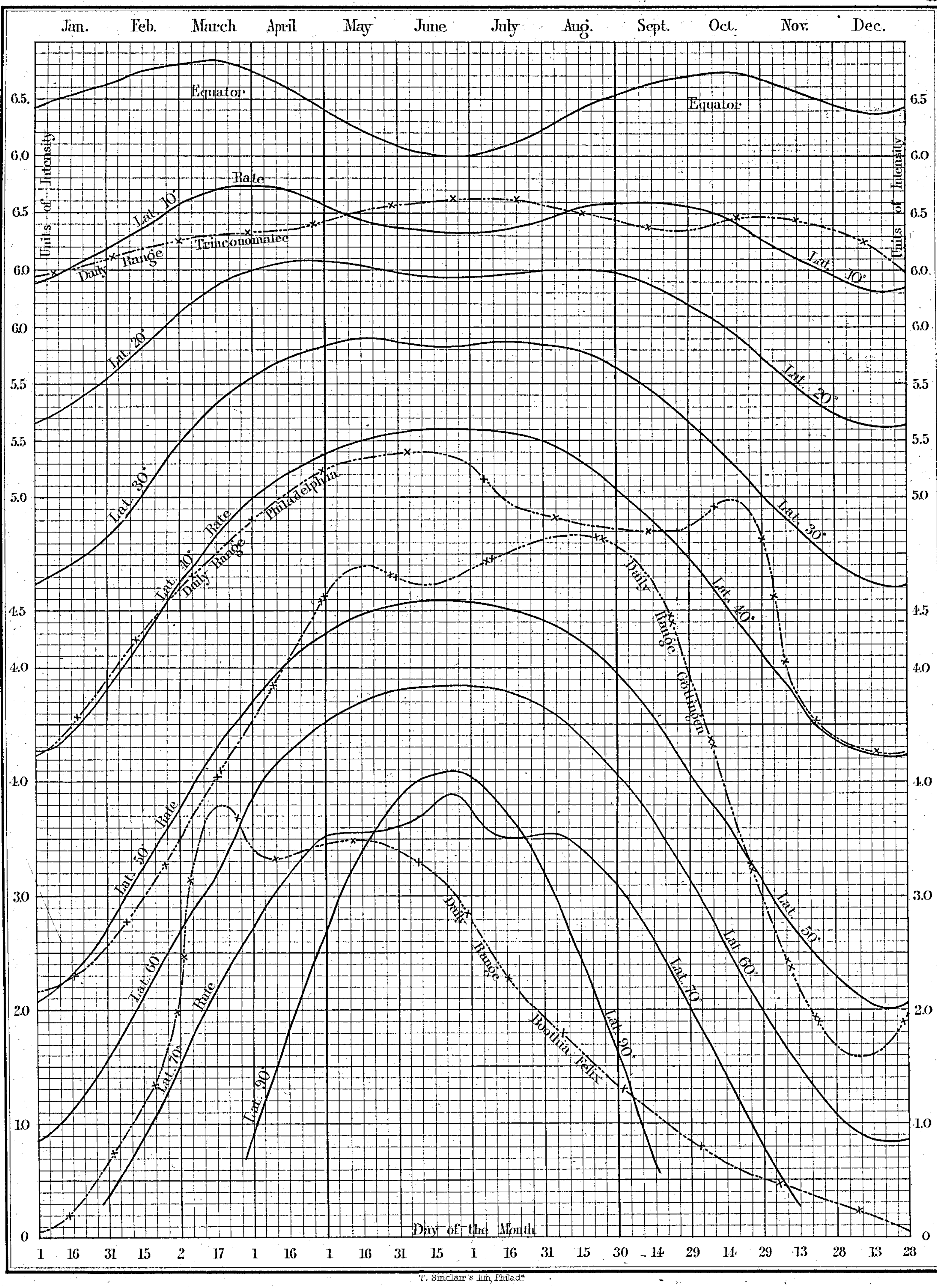
THE SUN'S DIURNAL INTENSITY.



T. Sinclair's lith, Phil^a

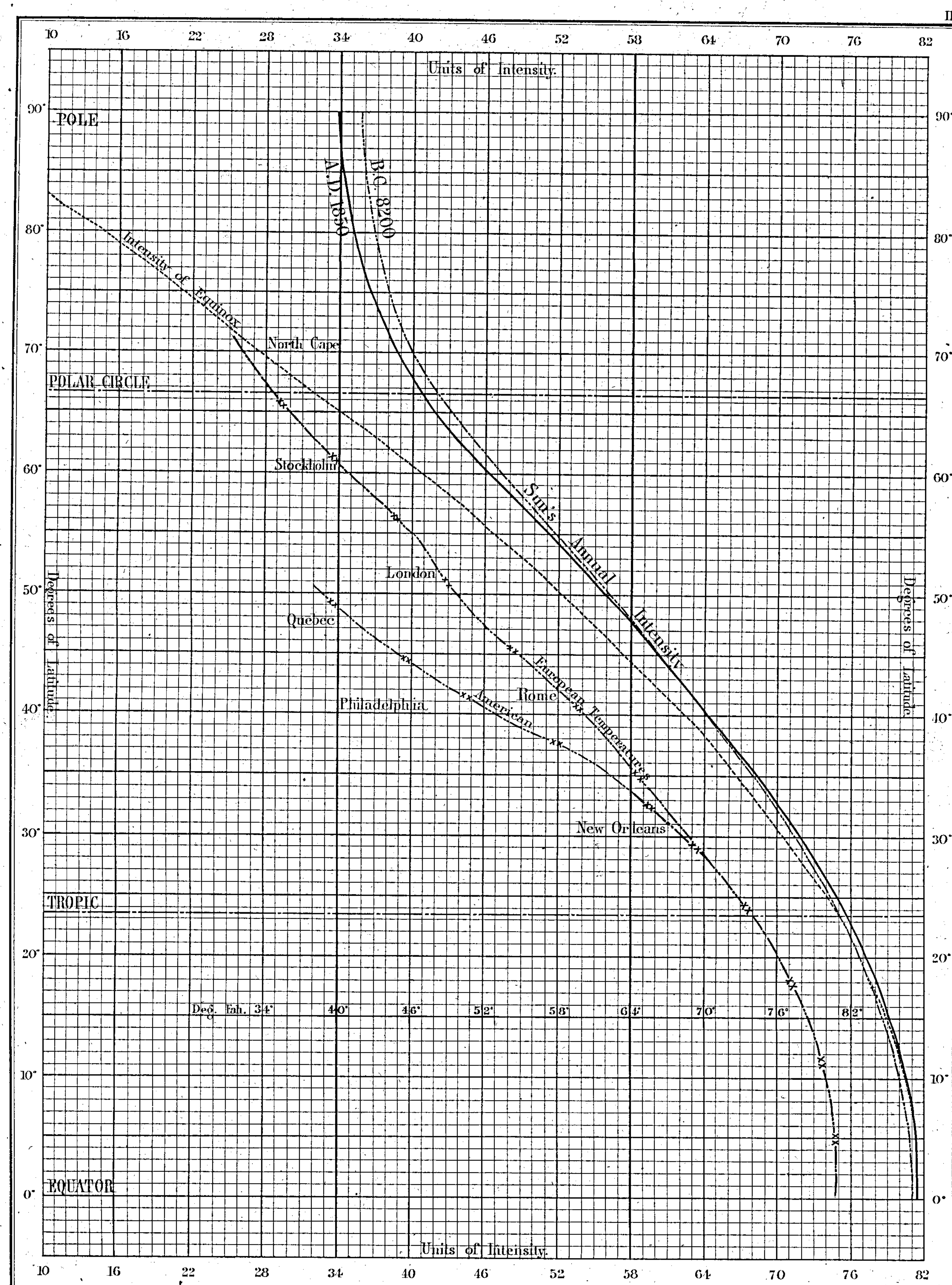
DAILY AVERAGE RATE PER HOUR OF THE SUN'S INTENSITY.

II.



T. Sinclair's lith, Philad^{a}.

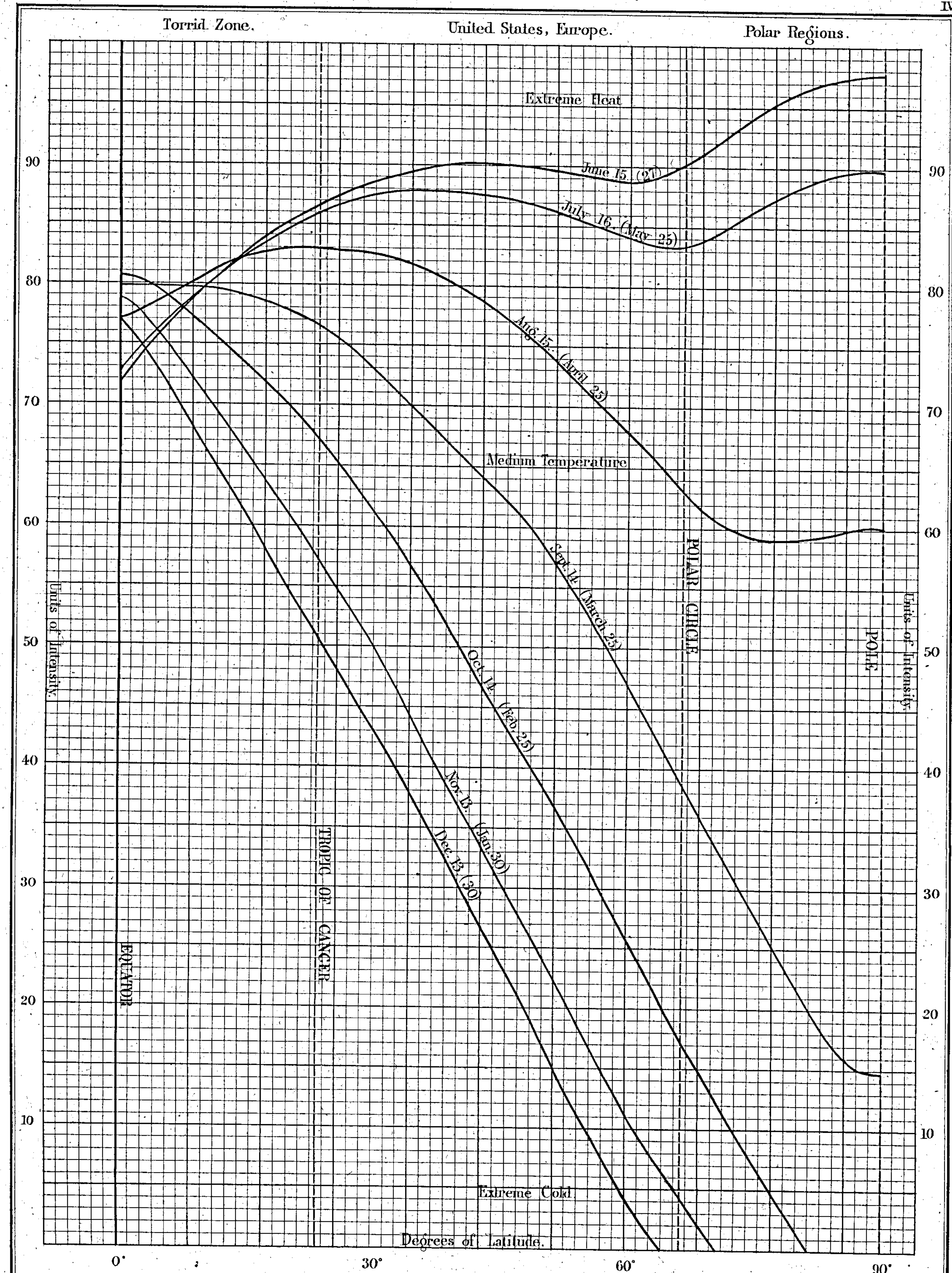
THE SUN'S ANNUAL INTENSITY.



T. Sinclair's lith, Phil^a.

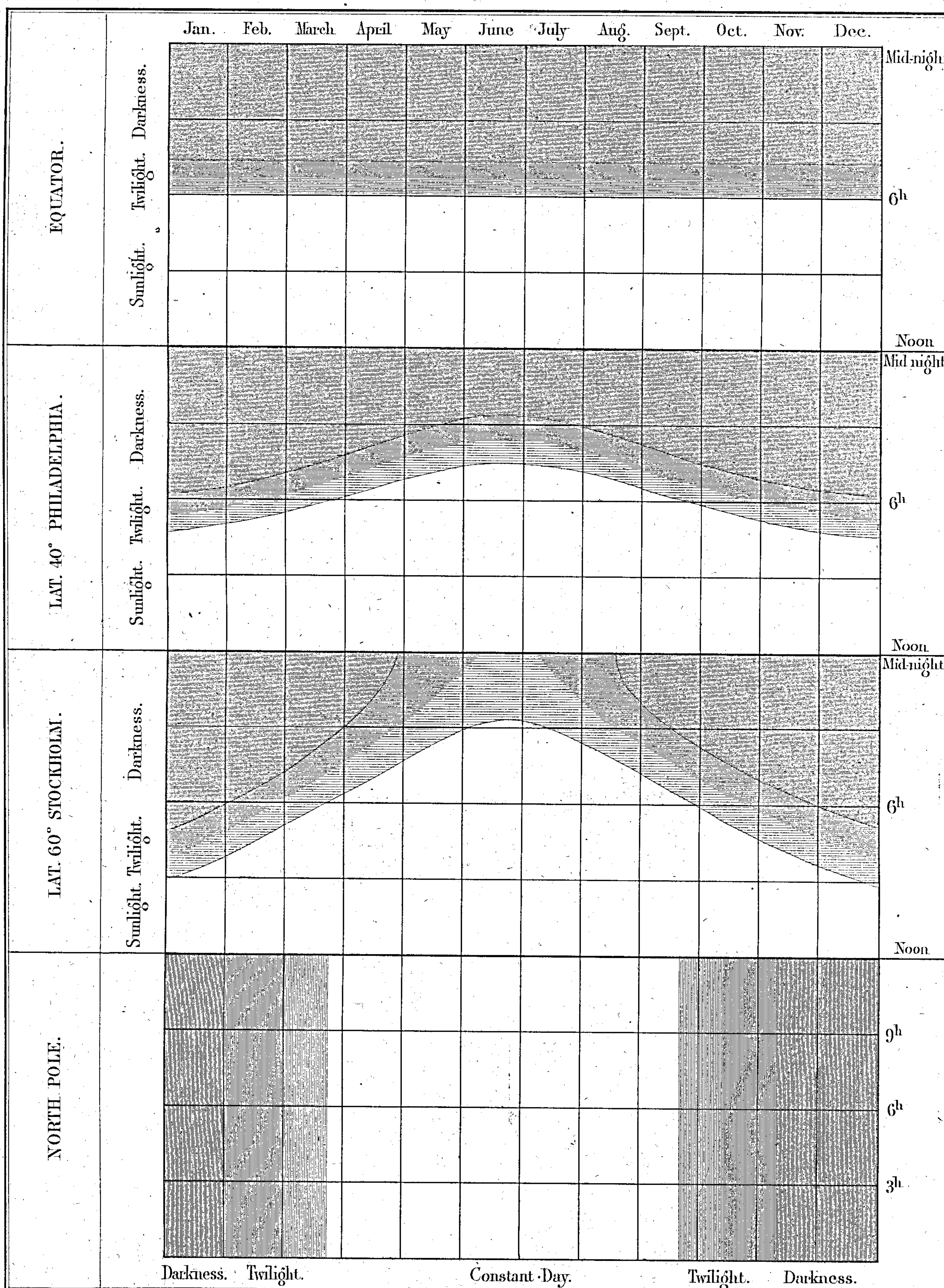
THE SUN'S DIURNAL INTENSITY ALONG THE MERIDIAN,

AT INTERVALS OF THIRTY DAYS.



T. Sinclair's lith, Philad[a].

DURATION OF SUNLIGHT, TWILIGHT AND DARKNESS.



T. Sinclair's lith. Phila.